The Tools of Macroeconomic Theory

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Chapter 1

Dynamic Stochastic General Equilibrium Models

1.1 A Static General Equilibrium Model

Here offer a brief review of the static general equilibrium framework and welfare theorems that facilitate our analysis. Whenever we consider a general equilibrium model, we must specify the following components:

Commodities What kinds of goods are traded in the economy?
In our static GE model, we assume there are M goods.

Preferences What are the agents in the economy, and how do they value commodities?
In our static GE model, we assume that there are I agents and that the commodity space of consumer i is given as the closed and convex set $X^i \subset \mathbb{R}^M$, which collects the vectors $x^i = (x_1^i, \cdots, x_M^i)$ representing how much of each good consumer i consumes. The set of all agents is $\mathcal{I} = \{1, \cdots, I\}$. We assume that consumer i's preferences admit a utility function representation via the function $u_i : X_i \to \mathbb{R}$.
We define $X \subset \mathbb{R}^{MI}$ as the Cartesian product of $X^1, \cdots, X^I$.

Technologies How are goods produced?
In our static GE model, we assume that there are J firms, where the jth

firm has production set $Y^j \subset \mathbb{R}^M$. The set of all firms is $\mathcal{J} = \{1, \cdots, J\}$. Normally, we assume that each production set is characterized by a transformation function $F^j : \mathbb{R}^M \to \mathbb{R}$ such that

$$Y^j = \{y \in \mathbb{R}^M \mid F_j(y) \leq 0\},$$

where $y \in \partial Y^j$ if and only if $F_j(y) = 0$. We assume that each F_j is convex, so that Y^j becomes a closed and convex set.
As above, we define $Y \subset \mathbb{R}^{MJ}$ as the Cartesian product of $Y^1, \cdots, Y^J$.
Consumer i owns the share θ_{ij} of firm j; naturally, $\sum_{i=1}^{I} \theta_{ij} = 1$ for any $1 \leq j \leq J$.

Endowments What resources do agents possess at the outset?
We assume that consumer i has endowment vector $e^i = (e^i_1, \cdots, e^i_M)$ of goods 1 to M. For notational simplicity, define

$$\underline{e}_k = \sum_{i=1}^{I} e^i_k,$$

that is, $\underline{e}_k$ is the total endowment of good k. Similarly, we define

$$\underline{e} = \begin{pmatrix} \underline{e}_1 \\ \vdots \\ \underline{e}_M \end{pmatrix} = \sum_{i=1}^{I} \begin{pmatrix} e^i_1 \\ \vdots \\ e^i_M \end{pmatrix}.$$

An allocation is defined as a vector

$$(x, y) = (x^1, \cdots, x^I, y^1, \cdots, y^J) \in X \times Y.$$

We say that the allocation (x, y) is feasible if, for any good $1 \leq k \leq M$,

$$\sum_{i \in \mathcal{I}} x^i_k \leq \sum_{i \in \mathcal{I}} e^i_k + \sum_{j \in \mathcal{J}} y^j_k,$$

that is, the total amount of good k consumed is less than or equal to the total amount

of good k in the economy, either given from the outset in the form of endowments or produced. Collecting all these inequalities together, we can see that $(x, y) \in X \times Y$ is feasible if

$$\sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j.$$

1.1.1 Pareto Optimality and the Social Planner's Problem

An allocation $(x, y) \in X \times Y$ is said to be Pareto efficient if:

1) It is a feasible allocation such that

2) It cannot be Pareto improved upon, that is, there does not exist another feasible allocation $(\hat{x}, \hat{y}) \in X \times Y$ such that

$$u_i(\hat{x}^i) \geq u_i(x^i)$$

for any $1 \leq i \leq I$, with at least one inequality being strict.

We can immediately see that a Pareto optimal allocation must satisfy the resource constraint with an equality if preferences are strictly monotonic.

Lemma Suppose preferences are strictly monotonic and that $(x, y) \in X \times Y$ is a Pareto efficient allocation. Then,

$$\sum_{i \in \mathcal{I}} x^i = \underline{e} + \sum_{j \in \mathcal{J}} y^j.$$

Proof) Suppose that

$$\sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j,$$

where at least one of the inequalities is strict. Assume without loss of generality

that

$$\sum_{i\in\mathcal{I}} x_1^i < \underline{e}_1 + \sum_{j\in\mathcal{J}} y_1^j.$$

Then, letting

$$\epsilon = \underline{e}_1 + \sum_{j\in\mathcal{J}} y_1^j - \sum_{i\in\mathcal{I}} x_1^i > 0,$$

define

$$\begin{aligned} \hat{x}^i &= x^i \quad \text{for any } 2 \le i \le I \\ \hat{x}_k^1 &= x_k^1 \quad \text{for any } 2 \le k \le M, \quad \hat{x}_1^1 = x_1^1 + \epsilon \end{aligned}$$

and $\hat{y} = y$. Then, $(\hat{x}, \hat{y}) \in X \times Y$ and

$$\sum_{i\in\mathcal{I}} \hat{x}_k^i \le \underline{e}_k + \sum_{j\in\mathcal{J}} \hat{y}_k^j$$

for any $2 \le k \le M$. As for the first good,

$$\sum_{i\in\mathcal{I}} \hat{x}_1^i = \sum_{i\in\mathcal{I}} x_1^i + \epsilon = \underline{e}_1 + \sum_{j\in\mathcal{J}} \hat{y}_1^j,$$

so $(\hat{x}, \hat{y})$ is a feasible allocation. In addition, $u_i(\hat{x}^i) = u_i(x^i)$ for $2 \le i \le I$ and $u_i(\hat{x}^1) > u_1(x^1)$ because of the strict monotonicity of preferences, so that $(\hat{x}, \hat{y})$ constitutes a Pareto improvement upon (x, y). This contradicts our initial assumption that (x, y) is a Pareto optimal allocation, so it must be the case that

$$\sum_{i\in\mathcal{I}} x^i = \underline{e} + \sum_{j\in\mathcal{J}} y^j.$$

Q.E.D.

We can also think of social welfare not in terms of Pareto optimality, but in terms of a social welfare function $W : X \to \mathbb{R}$, where W is a function of consumption profiles

$(x^1, \cdots, x^I) \in X$. Usually, we consider welfare functions $W : X \to \mathbb{R}$ defined as

$$W(x) = \sum_{i \in \mathcal{I}} \lambda_i \cdot u_i(x^i),$$

for any $x \in X$ and some Pareto weights $\lambda_1, \cdots, \lambda_I \in \mathbb{R}_{++}$. The concavity of the utility functions (or equivalently, decreasing marginal utilities) ensures that maximization of this type of social welfare function ensures some sort of equality; this will be seen shortly.

The social planner's problem (SPP) is given as the following:

$$\begin{aligned} \max_{(x,y) \in X \times Y} \quad & W(x) = \sum_{i \in \mathcal{I}} \lambda_i \cdot u_i(x^i) \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j, \end{aligned}$$

that is, he wants to find the feasible allocation that maximizes the social welfare function. The remarkable feature of the SPP is that, for any appropriate choice of weights $\lambda_1, \cdots, \lambda_I$, any Pareto optimal allocation can be achieved as a solution to the SPP.

Theorem (Equivalence between SPP and PO)
Given some weights $\lambda_1, \cdots, \lambda_I \in \mathbb{R}_{++}$, let $(x^*, y^*) \in X \times Y$ be a solution to the SPP

$$\begin{aligned} \max_{(x,y) \in X \times Y} \quad & W(x) = \sum_{i \in \mathcal{I}} \lambda_i \cdot u_i(x^i) \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j. \end{aligned}$$

Then, $(x^*, y^*) \in X \times Y$ is Pareto efficient.

Conversely, let $(x^*, y^*) \in X \times Y$ be a Pareto efficient allocation. If, for any $i \in \mathcal{I}$,

1) x^{i*} lies in the interior of X^i,

2) u_i is strictly monotonic,

3) u_i is strictly quasiconcave, and

4) u_i is continuously differentiable on the interior of X^i with non-zero gradients,

then there exist $\lambda_1, \cdots, \lambda_I \in \mathbb{R}_+$ not all equal to 0 such that (x^*, y^*) is a solution to the above SPP.

Proof) Let $(x^*, y^*) \in X \times Y$ be a solution to the SPP; by the formulation of the problem itself, (x^*, y^*) is feasible. Suppose that $(\hat{x}, \hat{y}) \in X \times Y$ is a feasible allocation that Pareto improves upon (x^*, y^*). Then,

$$\sum_{i \in \mathcal{I}} \hat{x}^i \leq \underline{e} + \sum_{j \in \mathcal{J}} \hat{y}^j$$

and $u_i(\hat{x}^i) \geq u_i(x^{i*})$ for any $i \in \mathcal{I}$, with at least one inequality being strict. Due to the positivity of the weights $\lambda_1, \cdots, \lambda_I$, this implies that

$$W(\hat{x}) > W(x^*).$$

Thus, $(\hat{x}, \hat{y})$ satisfies the constraints of the SPP that yields a higher value of the welfare function than (x^*, y^*). This contradicts our assumption that (x^*, y^*) is a solution to the SPP, so it must be the case that (x^*, y^*) is Pareto efficient.

Conversely, suppose that (x^*, y^*) is a Pareto efficient allocation; by definition, it is feasible and thus satisfies the constraints of any SPP. In addition, suppose the constraints on the preferences stated in the statement of the theorem are satisfied, so that each u_i is strictly monotonic, strictly quasiconcave, and continuously differentiable with non-zero gradients.

Consider the problem

$$\begin{aligned} \max_{x \in X} \quad & u_1(x^1) \\ \text{subject to} \quad & u_i(x^i) \geq \underline{u}_i := u_i(x^{i*}) \quad \text{for any } 2 \leq i \leq I, \\ & \sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^{j*}. \end{aligned}$$

Suppose x^* is not a solution to this problem. Then, there exists a $\hat{x} \in X$ such

that

$$u_i(\hat{x}^i) \geq u_i(x^{i*})$$

for any $2 \leq i \leq I$,

$$\sum_{i \in \mathcal{I}} \hat{x}^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^{j*}$$

and $u_1(\hat{x}^1) > u_1(x^{1*})$. In other words, $(\hat{x}, y^*)$ is a feasible allocation that Pareto improves on (x^*, y^*), which contradicts the Pareto optimality of (x^*, y^*). Therefore, x^* must be a solution to the above problem.

By assumption, x^* is an interior solution to the above problem, so by the KKT theorem, there exist $\lambda_2^*, \cdots, \lambda_I^* \in \mathbb{R}_+$ and $\eta^* \in \mathbb{R}_+^M$ such that

$$\lambda_i^* \cdot \boldsymbol{\nabla} u_i(x^{i*}) = \eta^* \in \mathbb{R}_+^M$$

for $1 \leq i \leq I$, where $\lambda_1^* = 1$. Since the gradients of u_1 are non-zero valued, η^* is non-zero.

By the strict quasiconcavity of each $u_1, \cdots, u_I$, the welfare function $W : X \to \mathbb{R}$ defined as

$$W(x) = \sum_{i \in \mathcal{I}} \lambda_i^* \cdot u_i(x^i)$$

for any $x \in X$ is also strictly quasiconcave. Consider the SPP

$$\begin{aligned} \max_{(x,y) \in X \times Y} \quad & W(x) = \sum_{i \in \mathcal{I}} \lambda_i^* \cdot u_i(x^i) \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j. \end{aligned}$$

Suppose that (x^*, y^*) is not a solution to the above SPP; then, there exists a feasible allocation $(\hat{x}, \hat{y})$ such that

$$W(\hat{x}) = \sum_{i \in \mathcal{I}} \lambda_i^* \cdot u_i(\hat{x}^i) > W(x^*) = \sum_{i \in \mathcal{I}} \lambda_i^* \cdot u_i(x^{i*}).$$

By the strict quasiconcavity of W,

$$\boldsymbol{\nabla} W(x^*)'(\hat{x} - x^*) > 0,$$

where

$$\boldsymbol{\nabla} W(x^*) = \sum_{i \in \mathcal{I}} \lambda_i^* \cdot \boldsymbol{\nabla} u_i(x^{i*}) = I \cdot \eta^* \in \mathbb{R}_+^M.$$

Since η^* is non-zero, neither is $\boldsymbol{\nabla} W(x^*)$, so that $\hat{x} \geq x^*$ with at least one inequality being strict. By the strict monotonicity of preferences, this means that $u_i(\hat{x}^i) \geq u_i(x^{i*})$ for any $i \in \mathcal{I}$ with at least one inequality being strict. This contradicts the fact that (x^*, y^*) is a Pareto efficient allocation, so it must be the case that (x^*, y^*) is a solution to the above SPP.

Q.E.D.

When we characterize production sets via transformation functions, the SPP can be rewritten as

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}^{M(I+J)}} \quad & W(x) = \sum_{i \in \mathcal{I}} \lambda_i \cdot u_i(x^i) \\ \text{subject to} \quad & x \in X, \quad F_j(y^j) \leq 0 \quad \text{for any } j \in \mathcal{J} \\ & \sum_{i \in \mathcal{I}} x^i \leq \underline{e} + \sum_{j \in \mathcal{J}} y^j. \end{aligned}$$

The Lagrangian of this problem is

$$\mathcal{L} = \sum_{i \in \mathcal{I}} \lambda_i \cdot u_i(x^i) - \sum_{j \in \mathcal{J}} \mu_j \cdot F_j(y^j) + \eta' \left(\sum_{j \in \mathcal{J}} y^j + \underline{e} - \sum_{i \in \mathcal{I}} x^i \right),$$

where $\mu_1, \cdots, \mu_J \in \mathbb{R}$ and $\eta \in \mathbb{R}^M$. Let (x^*, y^*) be a solution to this problem. Under differentiability and regularity assumptions, (x^*, y^*) satisfies the KKT first order conditions

$$\frac{\partial \mathcal{L}}{\partial x^i} = \lambda_i \cdot \boldsymbol{\nabla} u_i(x^{i*}) - \eta = \mathbf{0} \quad \text{for any } i \in \mathcal{I}$$

$$\frac{\partial \mathcal{L}}{\partial y^j} = -\mu_j \cdot \boldsymbol{\nabla} F_j(y^{j*}) + \eta = \mathbf{0} \quad \text{for any } j \in \mathcal{J}.$$

In other words,

$$\lambda_i \cdot \boldsymbol{\nabla} u_i(x^{i*}) = \lambda_{i'} \cdot \boldsymbol{\nabla} u_{i'}(x^{i'*})$$

for any $i, i' \in \mathcal{I}$ and

$$\mu_j \cdot \boldsymbol{\nabla} F_j(y^{j*}) = \mu_{j'} \cdot \boldsymbol{\nabla} F_{j'}(y^{j'*})$$

for any $j, j' \in \mathcal{J}$. The first condition tells us that the weighted marginal utilities are the same for all goods across all consumers; this is exactly how the utilitarian social welfare function ensures a degree of equality among consumers.

1.1.2 Competitive Equilibrium

A competitive equilibrium refers to a collection of prices $p^* \in \mathbb{R}^M$ and an allocation $(x^*, y^*) \in X \times Y$ satisfying the following conditions:

1) **Profit Maximization**
   For any $j \in \mathcal{J}$,

$$\langle p^*, y^{j*} \rangle = \max_{y^j \in Y^j} \langle p^*, y^j \rangle.$$

2) **Utility Maximization**
   For any $i \in \mathcal{I}$, x^{i*} is a solution to the problem

$$\begin{aligned} \max_{x^i \in X^i} \quad & u_i(x^i) \\ \text{subject to} \quad & \langle p^*, x^i \rangle \leq \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle. \end{aligned}$$

3) **Market Clearing**

The allocation (x^*, y^*) is feasible:

$$\sum_{i \in \mathcal{I}} x^{i*} \leq \underline{e} + \sum_{j \in \mathcal{J}} y^{j*}.$$

Putting aside the problem of the existence of competitive equilibria for a moment, we focus on the relationship between Pareto optimality and competitive equilibria, specifically the two fundamental theorems of welfare.

The first fundamental theorem of welfare states that any allocation achieved by competitive equilibrium is Pareto efficient. This is a useful result, since it indicates that we can find a competitive equilibrium allocation via the SPP, given certain assumptions on agents' utilities and well-chosen Pareto weights.

Lemma (Walras' Law)
Suppose agents' preferences are locally non-satiated, and let $x^{i*} \in X^i$ be a solution to agent i's utility maximization problem under the non-zero price vector $p \in \mathbb{R}^M_+ \setminus \{\mathbf{0}\}$ and income $m \geq 0$:

$$\begin{aligned} \max_{x^i \in X^i} \quad & u_i(x^i) \\ \text{subject to} \quad & \langle p, x^i \rangle \leq m. \end{aligned}$$

Then, the budget constraint holds as an equality at x^{i*}, that is, $\langle p, x^{i*} \rangle = m$.

Proof) Suppose

$$\langle p, x^{i*} \rangle < m.$$

Define

$$\epsilon = m - \langle p, x^{i*} \rangle > 0.$$

By local non-satiation, there exists an $\hat{x}^i \in X^i$ such that

$$\left| \hat{x}^i - x^{i*} \right| < \frac{\epsilon}{|p|}$$

and

$$u_i(\hat{x}^i) > u_i(x^{i*}).$$

Since

$$\begin{aligned}\langle p, \hat{x}^i \rangle &= \langle p, x^{i*} \rangle + \langle p, \hat{x}^i - x^{i*} \rangle \\ &\leq \langle p, x^{i*} \rangle + |p| \cdot \left| \hat{x}^i - x^{i*} \right| \\ &< \langle p, x^{i*} \rangle + \epsilon = m,\end{aligned}$$

we can see that $\hat{x}^i \in X^i$ satisfies the budget constraint but yields a higher value for u_i than x^{i*}. This contradicts the fact that x^{i*} is a solution to the utility maximization problem for agent i, and as such the budget constraint must be binding at x^{i*}.

Q.E.D.

Theorem (First Fundamental Theorem of Welfare)
Suppose agents' preferences are locally non-satiated. In this case, if $(p^*, (x^*, y^*)) \in \left(\mathbb{R}^M_+ \setminus \{\mathbf{0}\}\right) \times (X \times Y)$ is a competitive equilibrium, then, (x^*, y^*) is Pareto optimal.

Proof) Note that (x^*, y^*) is feasible by the market clearing condition of competitive equilibria. Suppose (x^*, y^*) is not Pareto efficient, so that there exists a feasible allocation $(\hat{x}, \hat{y}) \in X \times Y$ that Pareto improves on (x^*, y^*). By definition, $u_i(\hat{x}^i) \geq u_i(x^{i*})$ for any $i \in \mathcal{I}$, with at least one inequality being strict.

Choose some $i \in \mathcal{I}$. Suppose $u_i(\hat{x}^i) = u_i(x^{i*})$, and that

$$\langle p^*, \hat{x}^i \rangle < \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle.$$

Then, $\hat{x}^i$ satisfies the budget constraint for agent i, so that $\hat{x}^i$ is also a solution to agent i's utility maximization problem. Then, by Walras' law, the budget constraint has to hold as an equality under $\hat{x}^i$, which contradicts our initial

assumption. Thus, it must be the case taht

$$\langle p^*, \hat{x}^i \rangle \geq \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle.$$

On the other hand, if $u_i(\hat{x}^i) > u_i(x^{i*})$, then $\hat{x}^i$ must satisfy

$$\langle p^*, \hat{x}^i \rangle > \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle,$$

since otherwise $\hat{x}^i$ satisfies the budget constraint under the price vector p^* and budget $m = \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle$ and yields a higher value of u_i than x^{i*}, which contradicts the fact that x^{i*} is a solution to the utility maximization problem.

Therefore, in any case

$$\langle p^*, \hat{x}^i \rangle \geq \langle p^*, e^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle,$$

where at least one inequality is strict. Summing up both sides with respect to i yields

$$\left\langle p^*, \sum_{i \in \mathcal{I}} \hat{x}^i \right\rangle > \langle p^*, \underline{e} \rangle + \left\langle p^*, \sum_{j \in \mathcal{J}} y^{j*} \right\rangle.$$

By the feasibility of $(\hat{x}, \hat{y})$, we also have

$$\sum_{i \in \mathcal{I}} \hat{x}^i \leq \underline{e} + \sum_{j \in \mathcal{J}} \hat{y}^j,$$

and since the prices in p^* are all non-negative, taking the inner product on both sides with respect to p^* preserves the inequality and reveals that

$$\left\langle p^*, \sum_{i \in \mathcal{I}} \hat{x}^i \right\rangle \leq \langle p^*, \underline{e} \rangle + \sum_{j \in \mathcal{J}} \left\langle p^*, \hat{y}^j \right\rangle.$$

Furthermore, because y^{j*} maximizes profits under the price vector p^*, we have

$$\langle p^*, \hat{y}^j \rangle \leq \langle p^*, y^{j*} \rangle$$

for each $j \in \mathcal{J}$. By implication,

$$\left\langle p^*, \sum_{i\in\mathcal{I}} \hat{x}^i \right\rangle \leq \langle p^*, \underline{e}\rangle + \sum_{j\in\mathcal{J}} \left\langle p^*, y^{j*} \right\rangle = \langle p^*, \underline{e}\rangle + \left\langle p^*, \sum_{j\in\mathcal{J}} y^{j*} \right\rangle.$$

We therefore have

$$\langle p^*, \underline{e}\rangle + \left\langle p^*, \sum_{j\in\mathcal{J}} y^{j*} \right\rangle < \langle p^*, \underline{e}\rangle + \left\langle p^*, \sum_{j\in\mathcal{J}} y^{j*} \right\rangle,$$

a contradiction. Thus, (x^*, y^*) is Pareto efficient.

Q.E.D.

The second fundamental theorem of welfare is a partial converse to the first. Specifically, it posits that, for any Pareto efficient allocation, there exists an appropriate redistribution of the endowments (what is meant by redistribution is discussed formally below) such that there exists a price vector under which this allocation is a competitive equilibrium. The general proof of the second welfare theorem involves using the existence of competitive equilibrium to find the desired price vector. We are not interested in taking a detour to prove the existence of competitive equilibria, however, so we present an alternate proof of the theorem under stronger assumptions laid out in Varian (1992).

Theorem (Second Fundamental Theorem of Welfare)
Suppose the following hold:

1) Agents' preferences are strictly monotonic, continuous, and convex.

2) $X^i = \mathbb{R}^M_+$ for any $i \in \mathcal{I}$.

3) Y^j is non-empty and convex for any $j \in \mathcal{J}$.

Let (x^*, y^*) be a Pareto efficient allocation such that $x^{i*} \in \mathbb{R}^M_{++}$ for any $i \in \mathcal{I}$. Then, there exist lump sum transfers of goods $T^1, \cdots, T^I \in \mathbb{R}^M$ such that $\sum_{i\in\mathcal{I}} T^i = \mathbf{0}$ and a non-zero price vector $p^* \in \mathbb{R}^M_+$ such that $(p^*, (x^*, y^*))$ is a competitive equilibrium.

Proof) By the assumptions on agents' preferences, each u_i is strictly monotonic, continuous and quasiconcave.

First, note by strict monotonicity that

$$\sum_{i\in\mathcal{I}} x^{i*} = \underline{e} + \sum_{j\in\mathcal{J}} y^{j*}.$$

Define the transfer for the ith consumer as

$$T^i = x^{i*} - e^i - \sum_{j\in\mathcal{J}} \theta_{ij}\cdot y^{j*},$$

so that the new endowment becomes

$$\omega^i = e^i + T^i = x^{i*} - \sum_{j\in\mathcal{J}} \theta_{ij}\cdot y^{j*}.$$

Since the market clearing condition holds as an equality, we can see that

$$\sum_{i\in\mathcal{I}} T^i = \mathbf{0}$$

by design. As such, the allocation (x^*, y^*) clears the market even under the new endowments $\omega = (\omega^1, \cdots, \omega^I)$:

$$\begin{aligned}\sum_{i\in\mathcal{I}} x^{i*} &= \sum_{i\in\mathcal{I}} e^i + \sum_{j\in\mathcal{J}} y^{j*} \\ &= \sum_{i\in\mathcal{I}} \omega^i - \sum_{i\in\mathcal{I}} T^i + \sum_{j\in\mathcal{J}} y^{j*} = \sum_{i\in\mathcal{I}} \omega^i + \sum_{j\in\mathcal{J}} y^{j*}.\end{aligned}$$

It remains to find a price vector $p^* \in \mathbb{R}^M_+$ under which (x^*, y^*) maximizes profits and utility.

In our proof, we obtain the equilibrium price vector as a vector perpendicular to some separating hyperplane. To this end, we take recourse to the (general) separating hyperplane theorem. For any $i \in \mathcal{I}$, define the set P_i as

$$P_i = \{x^i \in X^i \mid u_i(x^i) > u_i(x^{i*})\}.$$

Note that, for any $x, y \in X^i$ and $\lambda \in [0,1]$, the quasiconcavity of u_i implies

$$u_i(\lambda \cdot x + (1-\lambda) \cdot y) \geq \min(u_i(x), u_i(y)) > u_i(x^{i*}),$$

so $\lambda \cdot x + (1-\lambda) \cdot y \in P_i$; therefore, P_i is a convex set. Furthermore, it is non-empty because X^i is unbounded above and any vector in X^i whose coordinates are greater than those of x^{i*} are contained in P_i. Defining

$$P = \sum_{i \in \mathcal{I}} P_i = \left\{ \sum_{i \in \mathcal{I}} x^i \mid x^i \in P_i \ \forall i \in \mathcal{I} \right\},$$

P becomes a non-empty convex set itself because the sum of convex sets is convex.

Similarly, define

$$F = \sum_{j \in \mathcal{J}} Y^j + \underline{e} = \left\{ \underline{e} + \sum_{j \in \mathcal{J}} y^j \mid y^j \in Y^j \ j \in \mathcal{J} \right\}.$$

Each Y^j is a convex set by assumption, so its sum $\sum_{j \in \mathcal{J}} Y^j$ is convex, and since the translation of a convex set is still convex, F is a convex subset of $\mathbb{R}^M$. It is also non-empty because the production sets $Y^1, \cdots, Y^J$ are non-empty. It is worth noting at this point that, because

$$\sum_{i \in \mathcal{I}} x^{i*} = \underline{e} + \sum_{j \in \mathcal{J}} y^{j*},$$

the sum $\sum_{i \in \mathcal{I}} x^{i*}$ is also contained in F.

Note that F and P are disjoint sets; to see this, suppose

$$\sum_{i \in \mathcal{I}} x^i \in F$$

for some $x = (x^1, \cdots, x^I) \in P_1 \times \cdots \times P_I$. Then, there exist $y = (y^1, \cdots, y^J) \in Y$ such that

$$\sum_{i \in \mathcal{I}} x^i = \underline{e} + \sum_{j \in \mathcal{J}} y^j,$$

so that $(x, y) \in X \times Y$ is feasible. However, by design, $u_i(x^i) > u_i(x^{i*})$ for any

$i \in \mathcal{I}$, which contradicts the Pareto optimality of (x^*, y^*).

So far, we have constructed two disjoint subsets P and F of $\mathbb{R}^M$ that are non-empty and convex. By the separating hyperplane theorem, there exists a non-zero vector $p^* \in \mathbb{R}^M$ such that

$$\langle p^*, z \rangle \geq \langle p^*, w \rangle$$

for any $z \in P$ and $w \in F$. We show below that this p^* is exactly the price vector that we want:

Step 1: p^* lies on the positive orthant of $\mathbb{R}^M$

To see this, choose some index $1 \leq k \leq M$ and consider the consumption profile

$$\hat{x} = (\hat{x}^1, \cdots, \hat{x}^I)$$

defined as

$$\hat{x}^i = x^{i*} + \frac{1}{I} e_k$$

for any $i \in \mathcal{I}$, where e_k is the kth standard basis vector of $\mathbb{R}^M$. Then, by the strict monotonicity of each agent, we have

$$u_i(\hat{x}^i) > u_i(x^{i*})$$

and therefore $\hat{x}^i \in P_i$ for $i \in \mathcal{I}$. Therefore, $\sum_{i \in \mathcal{I}} \hat{x}^i \in P$ and we have

$$\left\langle p^*, \sum_{i \in \mathcal{I}} \hat{x}^i \right\rangle \geq \left\langle p^*, \sum_{i \in \mathcal{I}} x^{i*} \right\rangle,$$

which implies

$$0 \leq \langle p^*, e_k \rangle = p_k^*.$$

This holds for each $1 \leq k \leq M$, so we have $p^* \in \mathbb{R}_+^M$.

Step 2: Profit Maximization

Choose any $j \in \mathcal{J}$, $y^j \in Y^j$ and $\epsilon > 0$. Defining

$$\hat{x}^i = x^{i*} + \frac{\epsilon}{I} \cdot \iota \in X^i$$

for any $i \in \mathcal{I}$, where $\iota \in \mathbb{R}^M$ is a vector of ones, note that

$$\sum_{i \in \mathcal{I}} \hat{x}^i = \sum_{i \in \mathcal{I}} x^{i*} + \epsilon \cdot \iota = \underline{e} + \sum_{l \neq j} y^{l*} + y^{j*} + \epsilon \cdot \iota = \sum_{i \in \mathcal{I}} x^{i*} + \epsilon \cdot \iota.$$

By strict monotonicity, $\hat{x}^i \in P_i$ for any $i \in \mathcal{I}$, so

$$\underline{e} + \sum_{l \neq j} y^{l*} + y^{j*} + \epsilon \cdot \iota = \sum_{i \in \mathcal{I}} x^{i*} + \theta \cdot \iota \in P.$$

On the other hand,

$$\underline{e} + y^j + \sum_{l \neq j} y^{l*} \in F.$$

Therefore, by the construction of p^*, we have

$$\left\langle p^*,\, \underline{e} + \sum_{l \neq j} y^{l*} + y^{j*} + \epsilon \cdot \iota \right\rangle \geq \left\langle p^*,\, \underline{e} + y^j + \sum_{l \neq j} y^{l*} \right\rangle.$$

Canceling out all the overlapping terms, we are left with

$$\langle p^*, y^{j*} \rangle + \epsilon \cdot \left(\sum_{k=1}^{M} p_k^* \right) \geq \langle p^*, y^j \rangle.$$

Since $\sum_{k=1}^{M} p_k^* \geq 0$ and this holds for any $\epsilon > 0$, sending $\epsilon \to 0$ on both sides shows us that

$$\langle p^*, y^{j*} \rangle \geq \langle p^*, y^j \rangle.$$

This holds for any $y^j \in Y^j$, so it follows that

$$\langle p^*, y^{j*} \rangle = \max_{y^j \in Y^j} \langle p^*, y^j \rangle.$$

Step 3: Utility Maximization

Now choose some $i \in \mathcal{I}$, and suppose that

$$u_i(\hat{x}^i) > u_i(x^{i*})$$

for some $\hat{x}^i \in X^i$. By the continuity of u_i,

$$\lim_{\theta \uparrow 1} u_i\left(\theta \hat{x}^i\right) = u_i(\hat{x}^i),$$

so it follows that there exists some $\theta \in (0,1)$ such that

$$u_i(\theta \hat{x}^i) > u_i(x^{i*}).$$

Define

$$z^l = \begin{cases} x^{l*} + \frac{1-\theta}{I-1} \cdot \hat{x}^i & \text{if } l \neq i \\ \theta \hat{x}^i & \end{cases}.$$

Then, because $\hat{x}^i$ must have at least one positive coordinate (if $\hat{x}^i = \mathbf{0}$ then $u_i(\hat{x}^i) < u_i(x^{i*})$ by strict monotonicity), by strict monotonicity $z^l \in P_l$ for any $l \in \mathcal{I}$. Therefore,

$$\sum_{l \in \mathcal{I}} z^l = \sum_{l \neq i} x^{l*} + \hat{x}^i \in P,$$

so that

$$\left\langle p^*, \sum_{l \neq i} x^{l*} + \hat{x}^i \right\rangle \geq \left\langle p^*, \sum_{l \in \mathcal{I}} x^{l*} \right\rangle,$$

where we used the fact that $\sum_{l \in \mathcal{I}} x^{l*} \in F$. Canceling out all the common terms from both sides yields

$$\langle p^*, \hat{x}^i \rangle \geq \langle p^*, x^{i*} \rangle.$$

We want to show that this inequality holds as a strict inequality. Suppose that it holds as an equality, that is,

$$\langle p^*, \hat{x}^i \rangle = \langle p^*, x^{i*} \rangle,$$

which is a positive value because $p^* \in \mathbb{R}^M_+$ is non-zero and x^{i*} is in $\mathbb{R}^M_{++}$ by assumption.

We saw above that there exists a $\theta \in (0, 1)$ such that

$$u_i(\theta \hat{x}^i) > u_i(x^{i*});$$

repeating the entire process above shows us that

$$\theta \cdot \langle p^*, \hat{x}^i \rangle = \langle p^*, \theta \hat{x}^i \rangle \geq \langle p^*, x^{i*} \rangle.$$

Therefore,

$$\theta \cdot \langle p^*, \hat{x}^i \rangle \geq \langle p^*, \hat{x}^i \rangle > 0,$$

and dividing both sides by $\langle p^*, \hat{x}^i \rangle$ shows us that $\theta \geq 1$, a contradiction. It follows that we must have

$$\langle p^*, \hat{x}^i \rangle > \langle p^*, x^{i*} \rangle.$$

So far, we have shown that, for any $\hat{x}^i \in X^i$ such that

$$u_i(\hat{x}^i) > u_i(x^{i*}),$$

we must have

$$\langle p^*, \hat{x}^i \rangle > \langle p^*, x^{i*} \rangle = \langle p^*, \omega^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle.$$

The contrapositive tells us that, if $x^i \in X^i$ and

$$\langle p^*, x^i \rangle \leq \langle p^*, \omega^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle,$$

then

$$u_i(x^i) \leq u_i(x^{i*}).$$

Since x^{i*} trivially satisfies the inequality above, this proves that x^{i*} maximizes u_i subject to the constraint

$$\langle p^*, x^i \rangle \leq \langle p^*, \omega^i \rangle + \sum_{j \in \mathcal{J}} \theta_{ij} \cdot \langle p^*, y^{j*} \rangle.$$

Q.E.D.

1.2 Introducing Dynamics and Stochastics

Now we study general equilibrium model that are dynamic, that is, agents make decisions over time, and stochastic, that is, the model includes uncertainty about the future. In these types of model, different dates and states of nature are treated like different goods in the economy; then, we can treat a dynamic stochastic general equilibrium model as a special case of a static general equilibrium model.

1.2.1 Model Assumptions

To facilitate our analysis, we make some simplifying assumptions. First, we assume an endowment economy, so that each agent is endowed with a given endowment in every period and every possible state of nature. This just means that we are not interested at the moment in how consumers earn their income. Furthermore, we assume that there is only one consumption good; this assumption can easily be relaxed, but is introduced for the sake of notational brevity. Lastly, agents are assumed to be risk-averse when it comes to consuming across states of nature in a given time period. This assumption is incorporated in the shape of the instantaneous utility function.

Formally, our assumptions are given below:

Commodities There are I agents in the economy, as in the static model; $\mathcal{I} = \{1, \cdots, I\}$ is the set of individuals.

There is one consumption good in each period. The initial period is 0 and we assume discrete time, so that our time index is $\mathbb{N} = \{0, 1, 2, \cdots\}$.

In addition, we assume that there are exogenous stochastic shocks s_t that occur at each time t. These are shocks that affect all individuals, hence the independence on the index i; we can take them to represent systematic macroeconomic shocks. The set of all possible time t shocks is denoted by S_t.

The collection $s^t = (s_0, \cdots, s_t)$ denotes the history of stochastic shocks, and $\mathcal{S}^t$ is the set of all possible histories of shocks at time t.

Finally, $\pi_t : \mathcal{S}^t \to (0, 1)$ is a function such that $\pi_t(s^t)$ is the probability of observing the history s^t at time t. Naturally, $\sum_{s^t \in \mathcal{S}^t} \pi_t(s^t) = 1$.

We take as our commodity space the collection of all individual consumption over time and across states of nature, denoted by $\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$. Each $c_{it}(s^t)$ is a non-negative real number, that is, it takes values in $\mathbb{R}_+$.

Preferences We assume that consumer i's preferences admit a separable von-Neumann Morgenstern (v-NM) utility representation. In other words, there exists a Bernoulli index, or instantaneous utility function, $u_i : \mathbb{R}_+ \to \mathbb{R}$ that is bounded, and a discount factor $\beta \in (0,1)$, such that agent i's intertemporal utility from consuming the commodity stream $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is given as

$$U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) = \sum_{t=0}^{\infty}\beta^t \sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)u_i(c_{it}(s^t)).$$

The boundedness assumption is crucial for this utility function to be well-defined, since, if we let $M > 0$ satisfy $|u_i(c)| \leq M$ for any $c \in \mathbb{R}_+$,

$$\begin{aligned}\sum_{t=0}^{\infty}\left|\beta^t \sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)u_i(c_{it}(s^t))\right| &\leq M\cdot\left(\sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)\right)\\ &= M\cdot\sum_{t=0}^{\infty}\beta^t = \frac{M}{1-\beta} < +\infty,\end{aligned}$$

so that the series $U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$ converges to a real value.

Furthermore, agents are assumed to have positive but diminishing marginal utility; that is, $u_i'(c) > 0$ and $u_i''(c) < 0$ for any $c > 0$. In addition, we assume the Inada condition

$$\lim_{c\downarrow 0} u_i'(c) = +\infty.$$

Endowments As for production, we abstract for a moment and assume that individual i receives an endowment of $y_{it}(s^t)$ at time t given the history s^t.

In our model, we treat each $c_{it}(s^t)$ for any $t \in \mathbb{N}$, $s^t \in \mathcal{S}^t$ and $i \in \mathcal{I}$ as a separate good

that can be traded at the initial period. It is in this sense that the economy is complete in the Arrow-Debreu model. This reduces the dynamic stochastic general equilibrium model to a static general equilibrium model, and we should expect the welfare theorem, at least the first theorem, to hold in this context as well.

1.2.2 Revisiting Pareto Optimality and the SPP

An allocation is given as the collection $\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ of consumptions across all periods in time and across states for every consumer. We say this allocation is feasible if

$$\sum_{i\in\mathcal{I}} c_{it}(s^t) \leq \sum_{i\in\mathcal{I}} y_{it}(s^t).$$

Analogously to the static general equilibrium case, we can now define the concepts of the Pareto efficiency and competitive equilibrium. We say the allocation $\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is Pareto efficient if:

1) It is a feasible allocation that

2) Cannot be Pareto improved upon, that is, there does not exist another feasible allocation $\{\hat{c}_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ such that

$$U_i(\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) \geq U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$$

for any $i \in \mathcal{I}$, with at least one inequality being strict.

We can also consider the SPP in this context as well; for a given set of Pareto weights $\lambda_1, \cdots, \lambda_I > 0$, the SPP can be formulated as

$$\begin{aligned} \max_{\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad & \sum_{i\in\mathcal{I}} \lambda_i \cdot U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) \\ \text{subject to} \quad & \sum_{i\in\mathcal{I}} c_{it}(s^t) \leq \sum_{i\in\mathcal{I}} y_{it}(s^t) \quad \text{for any } t\in\mathbb{N}, s^t\in\mathcal{S}^t. \end{aligned}$$

The same relationship holds in the DSGE case compared to the static GE model; specifically, any solution to the SPP must be Pareto optimal.

Theorem (SPP is Pareto Optimal)
Let $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ be a solution to the SPP

$$\max_{\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad \sum_{i\in\mathcal{I}} \lambda_i \cdot U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}} c_{it}(s^t) \leq \sum_{i\in\mathcal{I}} y_{it}(s^t) \quad \text{for any } t\in\mathbb{N}, s^t\in\mathcal{S}^t$$

given some Pareto weights $\lambda_1,\cdots,\lambda_I > 0$. Then, $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is Pareto optimal.

Proof) Suppose $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is not Pareto optimal. Then, there exists a feasible allocation $\{\hat{c}_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ such that

$$U_i(\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) \geq U_i(\{c_{it}^*(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$$

for any $i\in\mathcal{I}$, with at least one inequality being strict. Then, because each λ_i is strictly positive, this implies that

$$\sum_{i\in\mathcal{I}} \lambda_i \cdot U_i(\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) > \sum_{i\in\mathcal{I}} \lambda_i \cdot U_i(\{c_{it}^*(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}).$$

Since $\{\hat{c}_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ satisfies the constraints of the SPP by its feasibility, this means that $\{\hat{c}_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is an allocation that satisfies the constraints of the SPP but impoves upon $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ in terms of weighted utility. This contradicts the fact that $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ solves the SPP, so $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ must be Pareto optimal.

Q.E.D.

The theorem tells us that we can study the properties of Pareto optimal allocations in the DSGE model by deriving necessary conditions that solutions to SPP must satisfy.

1.2.3 Arrow-Debreu Equilibrium: Time 0 Trading

When discussing competitive equilibrium, we must talk of the decisions made by individual consumers. There are two ways that individuals can make their decisions. Here we first think of decision making as a process of trading Arrow-Debreu securities at time 0, that is, they trade at time 0 their consumption at every possible time and state of nature, given the prices of the final good at each time and state of nature. A general equilibrium model in which agents can trade, at the initial period, claims to consumption of the good at any time and at any state of nature, is called an Arrow-Debreu model of general equilibrium.

In the Arrow-Debreu model, given that the good has price $p_t^0(s^t)$ at time t and the history $s^t \in \mathcal{S}^t$, agent i solves the utility maximization problem

$$\begin{aligned} \max_{\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad & \sum_{t=0}^{\infty}\beta^t \sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)u_i(c_{it}(s^t)) \\ \text{subject to} \quad & \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)c_{it}(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)y_{it}(s^t). \end{aligned}$$

We say that the sequence of prices $\{p_t^0(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ and allocation $\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ constitutes a competitive equilibrium if:

1) **Utility Maximization**
$\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is a solution to the utility maximization problem

$$\begin{aligned} \max_{\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad & \sum_{t=0}^{\infty}\beta^t \sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)u_i(c_{it}(s^t)) \\ \text{subject to} \quad & \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)c_{it}(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)y_{it}(s^t). \end{aligned}$$

for any $i \in \mathcal{I}$.

2) **Market Clearing**
$\{c_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is feasible, that is, for any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$,

$$\sum_{i\in\mathcal{I}} c_{it}(s^t) \leq \sum_{i\in\mathcal{I}} y_{it}(s^t).$$

We can now show that the first welfare theorem holds in this context, given appropriate assumptions on the price sequence $\{p_t^0(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$.

Lemma (Walras' Law for DSGE Models)
Let $i \in \mathcal{I}$ and $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ a solution to the maximization problem

$$\max_{\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad \sum_{t=0}^{\infty}\beta^t \sum_{s^t\in\mathcal{S}^t} \pi_t(s^t)u_i(c_{it}(s^t))$$

$$\text{subject to} \quad \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t),$$

where $\{p_t^0(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is a time 0 price process such that $p_t^0(s^t) \geq 0$ for any $t \in \mathbb{N}, s^t \in \mathcal{S}^t$, and the series

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t)$$

is convergent. Then, the budget constraint holds as an equality, that is,

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) = \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t).$$

Proof) Suppose that the budget constraint holds as a strict inequality, that is,

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) < \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t).$$

Then, we let

$$\epsilon = \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t) - \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) > 0,$$

where this value is well-defined because $\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t)$ converges. Con-

struct the consumption stream $\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ as

$$\hat{c}_{it}(s^t) = c_{it}(s^t) + \frac{2^{-t-1}}{|S^t|\left(p_t^0(s^t)+1\right)}\cdot\epsilon$$

for any $t\in\mathbb{N}$ and $s^t\in\mathcal{S}^t$. Then, because $u_i'(c)>0$ for any $c>0$ and thus u_i is strictly increasing on $\mathbb{R}_+$, we have

$$u_i(c_{it}(s^t)) < u_i(\hat{c}_{it}(s^t))$$

for any $t\in\mathbb{N}$ and $s^t\in\mathcal{S}^t$. It follows that

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}u_i(\hat{c}_{it}(s^t)) \geq \sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}u_i(c_{it}(s^t)).$$

Suppose the above holds as an equality. Since both terms converge, this implies that

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}\left(u_i(\hat{c}_{it}(s^t))-u_i(c_{it}(s^t))\right)=0.$$

Infinite series can be treated as integration over the set $\mathbb{N}$ with respect to the counting measure, and the integrand $t\mapsto\beta^t\sum_{s^t\in\mathcal{S}^t}\left(u_i(\hat{c}_{it}(s^t))-u_i(c_{it}(s^t))\right)$ is a non-negative function on $\mathbb{N}$, so by the vanishing property of non-negative functions we can see that

$$\beta^t\sum_{s^t\in\mathcal{S}^t}\left(u_i(\hat{c}_{it}(s^t))-u_i(c_{it}(s^t))\right)=0,$$

and in particular $u_i(\hat{c}_{it}(s^t))=u_i(c_{it}(s^t))$ for every $t\in\mathbb{N}$. However, this contradicts our construction of $\hat{c}_{it}(s^t)$, so the inequality must hold strictly, that is,

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}u_i(\hat{c}_{it}(s^t)) > \sum_{t=0}^{\infty}\beta^t\sum_{s^t\in\mathcal{S}^t}u_i(c_{it}(s^t)).$$

We can also show that $\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ satisfies the budget constraint. By de-

sign,

$$\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)\hat{c}_{it}(s^t) = \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) + \epsilon\frac{2^{-t-1}}{|S^t|}\cdot\sum_{s^t\in\mathcal{S}^t}\frac{p_t^0(s^t)}{p_t^0(s^t)+1}$$

$$\leq \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) + 2^{-t-1}\cdot\epsilon$$

for any $t\in\mathbb{N}$, so that

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)\hat{c}_{it}(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) + \epsilon\left(\sum_{t=0}^{\infty}2^{-t-1}\right)$$

$$= \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) + \epsilon = \sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t).$$

Therefore, $\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ satisfies the budget constraint but strictly increases utility compared to $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$, which contradicts the fact that $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is a solution to the utility maximization problem. It follows that the budget constraint should hold as an equality under $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$.

Q.E.D.

Theorem (First Fundamental Theorem of Welfare for DSGE Models) Suppose the prices $\{p_t^*(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ and allocation $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ constitute a competitive equilibrium of the Arrow-Debreu economy, and that the series

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^*(s^t)y_{it}(s^t)$$

converges. Then, $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is Pareto optimal.

Proof) Suppose that $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is not Pareto optimal. Then, there exists a feasible allocation $\{\hat{c}_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ that Pareto improves on $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$, that is,

$$U_i(\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}) \geq U_i(\{c_{it}^*(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$$

for any $i \in \mathcal{I}$, with at least one inequality being strict.

For any $i \in \mathcal{I}$, suppose the inequality holds as an equality. Suppose in addition that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \hat{c}_{it}(s^t) < \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) y_{it}(s^t).$$

Then, $\{\hat{c}_{it}(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ satisfies the budget constraint under $\{p_t^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$, and $U_i(\{\hat{c}_{it}(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}) = U_i(\{c_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t})$, so that $\{\hat{c}_{it}(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ is a solution to the utility maximization problem. However, the preceding lemma tells us that, in this case, the budget constraint must be binding at $\{\hat{c}_{it}(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$, which contradicts our initial supposition. Therefore, we must have

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \hat{c}_{it}(s^t) \geq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) y_{it}(s^t).$$

On the other hand, if the inequality holds a strict inequality, then we must have

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \hat{c}_{it}(s^t) > \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) y_{it}(s^t),$$

otherwise $\{\hat{c}_{it}(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ satisfies the budget constraint and improves utility over $\{c_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$, which contradicts the fact that $\{c_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ is utility maximizing.

Thus, for any $i \in \mathcal{I}$,

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \hat{c}_{it}(s^t) \geq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) y_{it}(s^t),$$

with at least one inequality being strict. It follows that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \left(\sum_{i \in \mathcal{I}} \hat{c}_{it}(s^t) \right) > \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^*(s^t) \left(\sum_{i \in \mathcal{I}} y_{it}(s^t) \right).$$

However, by the feasibility of $\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$,

$$\sum_{i\in\mathcal{I}}\hat{c}_{it}(s^t)\leq\sum_{i\in\mathcal{I}}y_{it}(s^t)$$

for any $t\in\mathbb{N}$ and $s^t\in\mathcal{S}^t$, which implies

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^*(s^t)\left(\sum_{i\in\mathcal{I}}\hat{c}_{it}(s^t)\right)\leq\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^*(s^t)\left(\sum_{i\in\mathcal{I}}y_{it}(s^t)\right).$$

This contradicts the inequality we just derived, so it must be the case that $\{c_{it}^*(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is Pareto optimal.

Q.E.D.

Note that, as in the static GE model, Walras' law plays a crucial role in the proof of the first welfare theorem. In turn, the assumption that

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)y_{it}(s^t)$$

is convergent is central to the proof of Walras' law. By the nth term test for the convergence of series, it must then be the case that

$$\lim_{t\to\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)y_{it}(s^t)=0,$$

that is, the expected monetary value of the endowments must disappear in the long run. This is referred to as the transversality or No-Ponzi condition, and it will be discussed in more depth in a later section.

1.2.4 Radner Equilibrium: Sequential Trading with Financial Assets

Now consider the case where agents make decisions sequentially instead of trading Arrow-Debreu securities at the initial period. This approach to agent decision making points out that, as opposed to the Arrow-Debreu model, in which agents can purchase claims at the initial period to deliver the consumption good at any time and at any state of nature, not every good can serve the wealth-transferring role. Instead, it is emphasized that there are certain assets in which agents can invest in at each time t to transfer wealth into the next period. This type of model, in which there are specific financial instruments that perform the wealth-transferring role, and agents sequentially choose their consumption and investment positions each period, is called the Radner model of general equilibrium.

The Flow Budget Constraint

In the Radner model, at any time t with realized history s^t, agents can choose to invest in assets that yields 1 unit of the consumption good at time $t+1$ depending on the state at time $t+1$. We denote by $a_{i,t+1}(s^t, s_{t+1})$ the amount agent i invests in the asset that, given the realization s^t, gives 1 unit of the consumption good at time $t+1$ when s_{t+1} is realized. The price of this asset at time t, given the realization s^t, is denoted by $Q_t(s_{t+1} \mid s^t)$.

Therefore, agent i must make two choices at time t. First, given the realization s^t of the states of nature up to time t, they must choose $c_{it}(s^t)$, that is, how much to consume. They must also choose $a_{i,t+1}(s^t, s_{t+1})$ for each $s_{t+1} \in \mathcal{S}_{t+1}$, that is how much to invest in each asset. The budget available to the agent at time t is their endowment $y_{it}(s^t)$ along with the return $a_{it}(s^t)$ of the asset that agent i invested in during the preceding period that pays off under the realization of state $s_t \in \mathcal{S}_t$. The flow budget constraint can then be written as

$$c_{it}(s^t) + \sum_{s_{t+1} \in \mathcal{S}_{t+1}} Q_t(s_{t+1} \mid s^t) \cdot a_{i,t+1}(s^t, s_{t+1}) \leq y_{it}(s^t) + a_{it}(s^t).$$

The Natural Borrowing Limit

Given the presence of borrowing in the model, without any restrictions it may be possible for agents to borrow indefinitely, that is, it is possible for them to run a Ponzi scheme. For instance, if an agent takes on an infinite amount of debt at the beginning of the economy, then they will have an infinite amount of income each period and thus will be able to consume infinitely. This is why we need, in addition to the budget constraint above, impose a borrowing constraint at each time t.

Consider a natural borrowing limit at time t given the realization of the history s^t. We want the agent to be able to pay back their debt in the long run. Thus, we would not lend the agent more than the sum of the present value of their income streams, that is, the total amount of resources available to the agent at time t given the history s^t. Since the asset prices in this context serve the role as discount factors, the present value of the income at time $\tau \geq t$ given the history s^τ, where the first t states are equal to s^t, is given as

$$\left(\prod_{h=0}^{\tau-1} Q_{t+h}(s_{t+h+1} \mid s^{t+h})\right) y_{i\tau}(s^\tau).$$

Thus, the natural borrowing limit becomes

$$\underline{a}_{it}(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau \in S^\tau \mid s^t} \left(\prod_{h=0}^{\tau-1} Q_{t+h}(s_{t+h+1} \mid s^{t+h})\right) y_{i\tau}(s^\tau),$$

where

$$S^\tau \mid s^t = \{(s'_0, \cdots, s'_\tau) \in S^\tau \mid (s'_0, \cdots, s'_t) = s^t\}.$$

The No-Ponzi scheme condition is now given as

$$a_{it}(s^t) \geq -\underline{a}_{it}(s^t), \quad \underline{a}_{it}(s^t) < +\infty$$

for any $i \in \mathcal{I}, t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$. The finiteness of $\underline{a}_{it}(s^t)$ is baked into the definition of the No-Ponzi scheme condition; otherwise, we would not have any borrowing limit at

all.

We now introduce a useful simplification. Since s_0 is known, $\mathcal{S}^t \mid s_0 = \mathcal{S}^t$ for any $t \in \mathbb{N}$ and we have

$$\underline{a}_{i0}(s_0) = \sum_{\tau=0}^{\infty} \sum_{s^\tau \in S^\tau} \left(\prod_{h=0}^{\tau-1} Q_{t+h}(s_{t+h+1} \mid s^{t+h}) \right) y_{i\tau}(s^\tau).$$

If $\underline{a}_{i0}(s_0) < +\infty$ for any $i \in \mathcal{I}$, since $\underline{a}_{it}(s^t)$ is simply one of the tail sums of $\underline{a}_{i0}(s_0)$ for any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$, it follows that $\underline{a}_{it}(s^t) < +\infty$ for any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$. It thus suffices to impose the restriction that the natural borrowing limit at time 0 is finite. Note the similarity of $\underline{a}_{i0}(s_0)$ to the series

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t),$$

the lifetime income of an agent in an Arrow-Debreu economy under the Arrow-Debreu security prices $\{p_t^0(s^t)\}_{t\in\mathbb{N}, s^t \in \mathcal{S}^t}$. Thus, we might think of the requirement that $\underline{a}_{i0}(s_0)$ be finite as an equivalent condition to the requirement that the above series converges. We show below that this indeed is the case.

Allocations and Equilibrium in a Radner Model

An allocation in a Radner model thus has two components, the consumption stream $\{c_{it}(s^t)\}_{i\in\mathcal{I}, t\in\mathbb{N}, s^t\in\mathcal{S}^t}$, and the stream of asset positions $\{a_{it}(s^t)\}_{i\in\mathcal{I}, t\in\mathbb{N}, s^t\in\mathcal{S}^t}$. The initial asset positions $\{a_{i0}(s_0)\}_{i\in\mathcal{I}}$ are equal to 0 if we assume the agents do not start out with any wealth.

Normalizing the price of the consumption good to 1,

- A stream of asset prices $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N}, (s^t, s_{t+1})\in\mathcal{S}^{t+1}}$
- An allocation of consumption goods $\{c_{it}^*(s^t)\}_{i\in\mathcal{I}, t\in\mathbb{N}, s^t\in\mathcal{S}^t}$
- A stream of asset positions $\{a_{it}^*(s^t)\}_{i\in\mathcal{I}, t\in\mathbb{N}, s^t\in\mathcal{S}^t}$

are together said to constitute a Radner equilibrium if:

1) **Utility Maximization**

For any $i \in \mathcal{I}$, $\{c_{it}^*(s^t), a_{it}^*(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ is a solution to the problem

$$\max_{\{c_{it}(s^t),a_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad \sum_{t=0}^{\infty}\beta^t \sum_{s^t\in\mathcal{S}^t}\pi_t(s^t)u_i(c_{it}(s^t))$$

$$\text{subject to} \quad c_{it}(s^t) + \sum_{s_{t+1}\in S_{t+1}} a_{i,t+1}(s^t, s_{t+1})\cdot Q_t(s_{t+1} \mid s^t) \le y_{it}(s^t) + a_{it}(s^t) \quad \text{and}$$

$$a_{it}(s^t) \ge -\underline{a}_{it}(s^t) \quad \text{for any } t \in \mathbb{N}, s^t \in \mathcal{S}^t,$$

$$\underline{a}_{i0}(s_0) < +\infty$$

2) **Market Clearing**

For any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$,

$$\sum_{i\in\mathcal{I}} c_{it}^*(s^t) \le \sum_{i\in\mathcal{I}} y_{it}(s^t) \qquad \text{(Goods Market Clears)}$$

$$\sum_{i\in\mathcal{I}} a_{it}^*(s^t) \le 0. \qquad \text{(Asset Market Clears)}$$

It turns out that, even if not every good performs the wealth transferring effect and we incorporate the financial market into the model, any resulting equilibrium is an Arrow-Debreu equilibrium and thus Pareto optimal, under some mild assumptions. EVen the converse holds; for any Arrow-Debreu equilibrium allocation, we can find a stream of assets and asset prices that support that allocation as a Radner equilibrium.

Theorem (Equivalence of Radner and Arrow-Debreu Equilibria)
Suppose that the asset prices $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N},(s^t,s_{t+1})\in\mathcal{S}^{t+1}}$, allocation of consumption goods $\{c^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$, and the stream of asset positions $\{a^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ constitute a Radner equilibrium. Then, there exist time 0 prices $\{p^0_t(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ for Arrow-Debreu securities such that $\{c^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is supported as an Arrow-Debreu equilibrium, where the series

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p^0_t(s^t)y_{it}(s^t)$$

converges.

Conversely, let the price process $\{p^0_t(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ for Arrow-Debreu securities and the allocation of consumption goods $\{c^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ constitute an Arrow-Debreu equilibrium, where $p^0_0(s_0)$ is normalized to 1. Suppose that the series

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p^0_t(s^t)y_{it}(s^t)$$

converges. Then, there exist asset prices $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N},(s^t,s_{t+1})\in\mathcal{S}^{t+1}}$ and a stream of asset positions $\{a^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ such that $\{c^*_{it}(s^t)\}_{i\in\mathcal{I},t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is supported as a Radner equilibrium, where $\underline{a}_{it}(s^t) < +\infty$ for any $i \in \mathcal{I}, t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$.

Proof) First, we introduce some terminology and concepts that will greatly simplify our proof.

1. **Corresponding Arrow-Debreu Prices and Asset Prices**
   Let $\{p^0_t(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ be a process of prices for Arrow-Debreu securities, and $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N},s^{t+1}\in\mathcal{S}^{t+1}}$ a process of asset prices. We say that these two processes are **correspond** to one another if $p^0_0(s_0) = 1$ and

   $$Q_t(s_{t+1} \mid s^t) = \frac{p^0_{t+1}(s^{t+1})}{p^0_t(s^t)},$$

   for any $t \in \mathbb{N}$ and $s^{t+1} \in \mathcal{S}^{t+1}$, or equivalently,

   $$p^0_t(s^t) = \prod_{\tau=0}^{t-1} Q_\tau(s_{\tau+1} \mid s^\tau),$$

for any $t \in N_+$ and $s^t \in \mathcal{S}^t$.

2. **Induced Asset Position Process**

   Let there be an asset price process $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N},s^{t+1}\in\mathcal{S}^{t+1}}$ and an allocation of consumption goods $\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$. Defining $\{p_t^0(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ as the corresponding prices for Arrow-Debreu securities, the induced asset position process $\{a_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is defined as $a_{i0}(s_0) = 0$ and

$$a_{it}(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau \in \mathcal{S}^\tau | s^t} \frac{p_\tau^0(s^\tau)}{p_t^0(s^t)} \left(c_{i\tau}(s^\tau) - y_{i\tau}(s^\tau)\right).$$

   for any $t \in N_+$ and $s^t \in \mathcal{S}^t$.

3. **Time 0 Maximization Problem**

   The time 0 maximization problem for agent i under the Arrow-Debreu price process $\{p_t^0(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}$ is the problem

$$\max_{\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad U_i(\{c_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t})$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t) c_{it}(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t) y_{it}(s^t).$$

4. **Sequential Maximization Problem**

   The sequential maximization problem for agent i under the asset price process $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N},s^{t+1}\in\mathcal{S}^{t+1}}$ is the problem

$$\max_{\{c_{it}(s^t),a_{it}(s^t)\}_{t\in\mathbb{N},s^t\in\mathcal{S}^t}} \quad \sum_{t=0}^{\infty} \beta^t \sum_{s^t\in\mathcal{S}^t} \pi_t(s^t) u_i(c_{it}(s^t))$$

$$\text{subject to} \quad c_{it}(s^t) + \sum_{s_{t+1} \in S_{t+1}} a_{i,t+1}(s^t, s_{t+1}) \cdot Q_t(s_{t+1} \mid s^t) \leq y_{it}(s^t) + a_{it}(s^t),$$

$$a_{it}(s^t) \geq -\underline{a}_{it}(s^t) \quad \text{for any } t \in \mathbb{N}, s^t \in \mathcal{S}^t$$

$$\underline{a}_{i0}(s_0) < +\infty.$$

We can now see that the following results hold:

i. **Time 0 Constraint Implies Sequential Constraint**
Let $\{p_t^0(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ be an Arrow-Debreu price process and $\{c_{it}(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ an allocation of consumption goods that satisfies the constraints in the time 0 maximization problem:

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) c_{it}(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t).$$

Suppose in addition that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t)$$

converges, so that $\sum_t \sum_{s^t} p_t^0(s^t) c_{it}(s^t)$ does as well.

Let $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N}, s^{t+1}\in\mathcal{S}^{t+1}}$ be the asset price process corresponding to $\{p_t^0(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$, and $\{a_{it}(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ the induced asset position process. The convergence of the two series above imply that the induced asset positions are also well-defined and real valued.

By construction, $a_{it}^*(s^t)$ satisfies the No-Ponzi scheme condition, and

$$a_{it}(s^t) = c_{it}(s^t) - y_{it}(s^t) + \sum_{s_{t+1} \in S_{t+1}} \frac{p_{t+1}^0(s^{t+1})}{p_t^0(s^t)} \cdot \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau | s^{t+1}} \frac{p_\tau^0(s^\tau)}{p_{t+1}^0(s^{t+1})} (c_{i\tau}(s^\tau) - y_{i\tau}(s^\tau))$$

$$= c_{it}(s^t) - y_{it}(s^t) + \sum_{s_{t+1} \in S_{t+1}} Q_t(s_{t+1} \mid s^t) a_{i,t+1}(s^t, s_{t+1})$$

for any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$. Therefore, $\{c_{it}(s^t), a_{it}(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ satisfies the constraints in agent i's sequential maximization problem under the asset price process $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N}, s^{t+1}\in\mathcal{S}^{t+1}}$.

ii. **Sequential Constraint Implies Time 0 Constraint**

Let $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N}, s^{t+1}\in\mathcal{S}^{t+1}}$ be an asset price process, $\{c_{it}(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ an allocation of consumption goods, and $\{a_{it}(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ a process of asset positions that satisfy the constraints in the sequential maximization problem:

$$c_{it}(s^t) + \sum_{s_{t+1}\in S_{t+1}} a_{i,t+1}(s^t, s_{t+1}) \cdot Q_t(s_{t+1} \mid s^t) \leq y_{it}(s^t) + a_{it}(s^t)$$

and $a_{it}(s^t) \geq -\underline{a}_{it}(s^t)$ for any $t \in \mathbb{N}, s^t \in \mathcal{S}^t$. Suppose in addition that $\underline{a}_{i0}(s_0) < +\infty$.

Let $\{p_t^0(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}$ be the Arrow-Debreu security price process corresponding to $\{Q_t(s_{t+1} \mid s^t)\}_{t\in\mathbb{N}, s^{t+1}\in\mathcal{S}^{t+1}}$. Multiplying both sides of the time t sequential constraint by $p_t^0(s^t)$ and summing up with respect to s^t yields

$$\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) + \sum_{s^{t+1}\in S^{t+1}} a_{i,t+1}(s^{t+1}) \cdot p_{t+1}^0(s^{t+1}) \leq \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t) + \sum_{s^t\in S^t} a_{it}(s^t) \cdot p_t^0(s^t).$$

Define

$$A_{i,t+1} = \sum_{s^{t+1}\in S^{t+1}} a_{i,t+1}(s^{t+1}) \cdot p_{t+1}^0(s^{t+1})$$

for any $t \in \mathbb{N}$ and let $A_{i0} = 0$, reflecting the initial wealth position. Then, we can see that

$$\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t) - \sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)c_{it}(s^t) \geq A_{i,t+1} - A_{it}.$$

for any $t \in \mathbb{N}$. Summing up with respect to t, we obtain a telescoping sum

on the right hand side:

$$\sum_{t=0}^{T}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)(y_{it}(s^t)-c_{it}(s^t))\geq A_{i,T+1}.$$

By the no-Ponzi condition $a_{it}(s^t)\geq -\underline{a}_{it}(s^t)$, where

$$\underline{a}_{it}(s^t)=\sum_{\tau=t}^{\infty}\sum_{s^\tau\in S^\tau|s^t}\frac{p_\tau^0(s^\tau)}{p_t^0(s^t)}\cdot y_{i\tau}(s^\tau),$$

we can see that

$$\left|a_{it}(s^t)\right|\leq\sum_{\tau=t}^{\infty}\sum_{s^\tau\in S^\tau|s^t}\frac{p_\tau^0(s^\tau)}{p_t^0(s^t)}\cdot y_{i\tau}(s^\tau).$$

By implication,

$$\begin{aligned}\left|\sum_{s^t\in S^t}p_t^0(s^t)\cdot a_{it}(s^t)\right|&\leq\sum_{s^t\in S^t}\left|p_t^0(s^t)\cdot a_{it}(s^t)\right|\\&\leq\sum_{\tau=t}^{\infty}\sum_{s^\tau\in S^\tau}p_\tau^0(s^\tau)\cdot y_{i\tau}(s^\tau).\end{aligned}$$

Taking $t\to\infty$, because the series $\sum_{\tau=0}^{\infty}\sum_{s^\tau\in S^\tau}p_\tau^0(s^\tau)\cdot y_{i\tau}(s^\tau)=\underline{a}_{i0}(s_0)$ is convergent, the right hand side goes to 0, and as such

$$\lim_{t\to\infty}\sum_{s^t\in S^t}p_t^0(s^t)\cdot a_{it}(s^t)=0.$$

It follows that

$$\lim_{T\to\infty}A_{i,T+1}=0.$$

Therefore,

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}p_t^0(s^t)(y_{it}(s^t)-c_{it}(s^t))\geq 0,$$

showing us that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) c_{it}(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t).$$

We are now able to prove the desired results. Recall that, at an Arrow-Debreu equilibrium (ADE), the consumption allocation solves each time 0 maximization problem under the given Arrow-Debreu price process and satisfies the goods market clearing condition. On the other hand, at a Radner equilibrium (RE), the consumption allocation-asset position process pair solves each sequential maximization problem under the given asset price process and satisfies both the goods market clearing condition and the asset market clearing condition.

ADE is RE

Suppose $p^0 = \{p_t^0(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ and $c^* = \{c_{it}^*(s^t)\}_{i \in \mathcal{I}, t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ constitute an Arrow-Debreu equilibrium such that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t)$$

converges. Let $Q = \{Q_t(s_{t+1} \mid s^t)\}_{t \in \mathbb{N}, s^{t+1} \in \mathcal{S}^{t+1}}$ be the corresponding asset price process and $a^* = \{a_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ the induced asset position process. For any $i \in \mathcal{I}$, denote

$$c_i^* = \{c_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$$
$$a_i^* = \{a_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}.$$

First, note that

$$\underline{a}_{i0}(s_0) = \sum_{\tau=0}^{\infty} \sum_{s^\tau \in S^\tau} \frac{p_\tau^0(s^\tau)}{p_t^0(s^t)} y_{i\tau}(s^\tau) < +\infty$$

because we assumed that the right hand side converges. Thus, the finiteness of the natural borrowing limits is established.

Since $\sum_t \sum_{s^t} p_t^0(s^t) y_{it}(s^t)$ converges, by result i above we can see that (c_i^*, a_i^*) satisfies the constraint of the sequential problem under the asset price process Q. On the other hand, if (c_i, a_i) is a collection of consumption allocations and asset positions of agent i that satisfies the constraints of the sequential problem under Q, then due to the finiteness of $\underline{a}_{i0}(s_0)$ result ii above shows us that (c_i, a_i) must also satisfy the constraints of the time 0 problem. Because c_i^* solves the time 0 maximization problem of agent i under the Arrow-Debreu price process p^0, it follows that $U_i(c_i^*) \geq U_i(c_i)$. Since this holds for any (c_i, a_i) satisfying the constraints of the sequential problem under Q, it must be that (c_i^*, a_i^*) solves agent i's sequential maximization problem.

Finally, the market clearing condition holds because

$$\sum_{i \in \mathcal{I}} c_{it}^*(s^t) \leq \sum_{i \in \mathcal{I}} y_{it}(s^t)$$

for any $t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$ by the definition of an Arrow-Debreu equilibrium, and

$$\sum_{i \in \mathcal{I}} a_{it}^*(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau \in \mathcal{S}^\tau | s^t} \frac{p_\tau^0(s^\tau)}{p_t^0(s^t)} \sum_{i \in \mathcal{I}} (c_{i\tau}^*(s^\tau) - y_{i\tau}(s^\tau)) \leq 0$$

by the goods market clearing condition. Therefore, Q, c^*, a^* constitute a RE.

RE is ADE

Let the asset prices $Q = \{Q_t(s_{t+1} \mid s^t)\}_{t \in \mathbb{N}, s^{t+1} \in \mathcal{S}^{t+1}}$, consumption allocation $c^* = \{c_{it}^*(s^t)\}_{i \in \mathcal{I}, t \in \mathbb{N}, s^t \in \mathcal{S}^t}$, and asset positions $a^* = \{a_{it}^*(s^t)\}_{i \in \mathcal{I}, t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ constitute a Radner equilibrium such that each natural borrowing limit is real valued. Let $p^0 = \{p_t^0(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$ be the Arrow-Debreu price process corresponding to Q. As above, we denote

$$c_i^* = \{c_{it}^*(s^t)\}_{t \in \mathbb{N}, s^t \in \mathcal{S}^t}$$

$$a_i^* = \{a_{it}^*(s^t)\}_{t\in\mathbb{N}, s^t\in\mathcal{S}^t}.$$

for any $i \in \mathcal{I}$.

First, note that

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t} p_t^0(s^t)y_{it}(s^t) = \underline{a}_{i0}(s_0) < +\infty$$

by assumption. Because $\underline{a}_{i0}(s_0) < +\infty$, from result ii above it follows that (c_i^*, a_i^*), which satisfies the constraints of the sequential problem under Q, must also satisfy the constraints of the time 0 problem under p^0.

Let c_i be an allocation of agent i's consumption goods that satisfies the time 0 constraint under p^0. Then, because $\sum_t \sum_{s^t} p_t^0(s^t)y_{it}(s^t)$ is convergent, by result i above, letting a_i be the asset position process induced by p^0 and c_i, the pair (c_i, a_i) satisfies the constraints of the sequential problem under Q. By the definition of Radner equilibrium, (c_i^*, a_i^*) is a solution to agent i's sequential maximization problem under Q, so we have $U_i(c_i^*) \geq U_i(c_i)$. This holds for any allocation c_i of consumption goods that satisfies the time 0 constraint under p^0, so it follows that c_i^* must solve the time 0 maximization problem under p^0.

Since the goods market clearing condition is satisfied by definition in an RE, it follows that p^0, c^* constitute an ADE.

Q.E.D.

The theorem above shows us that, under certain conditions, any allocation of consumption goods supported as an Arrow-Debreu equilibrium can be supported as a Radner equilibrium and vice versa. In light of the first welfare theorem for Arrow-Debreu equilibrium, it suggests that any allocation achieved in Radner equilibrium is also Pareto optimal.

As emphasized earlier, the crucial condition here is the requirement that

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} p_t^0(s^t) y_{it}(s^t) < +\infty,$$

which we saw above is equivalent to the requirement that the natural borrowing limits

$$\underline{a}_{it} = \sum_{\tau=t}^{\infty} \sum_{s^\tau \in \mathcal{S}^\tau | s^t} \left[\prod_{h=0}^{\tau-1} Q_{t+h}(s_{t+h+1} \mid s^{t+h}) \right] y_{i\tau}(s^\tau)$$

are always finite for any $i \in \mathcal{I}, t \in \mathbb{N}$ and $s^t \in \mathcal{S}^t$.

Chapter 2

Deterministic Dynamic Problems and Sequential Solutions

Given the general equilibrium framework studied in the previous chapter, here we start studying models that are deterministic, that is, there is no uncertainty, but which are nonetheless dynamic. A general sequential approach to solving deterministic dynamic optimization problems is addressed, and then the neoclassical growth model is discussed as an application of the machinery developed in the first section. Afterward, we study continuous time models and how they elucidate the evolution of macroeconomic variables.

2.1 Dynamic Optimization Problems in Discrete Time

We consider the following (deterministic) setup. Let (X, τ) be a topological space; this will be our state space, that is, the space in which our state variables take their values. Consider a bounded function $U : X \times X \to \mathbb{R}$, a discount factor $\beta \in (0, 1)$, and a correspondence $\Gamma : X \to X$ that is non-empty valued. Then, given the initial state $x_0 \in X$, we can consider the dynamic optimization problem

$$\max_{\{x_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

$$\text{subject to} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N}.$$

Note that the objective function is well-defined by the boundedness of U, and the constraints are well-defined because Γ is non-empty valued. A sequence $\{x_t\}_{t\in\mathbb{N}} \in X^{\mathbb{N}}$ is referred to as a plan. We say a plan $\{x_t\}_{t\in\mathbb{N}}$ is said to be feasible given the initial value $x_0 \in X$ if

$$x_{t+1} \in \Gamma(x_t)$$

for any $t \in \mathbb{N}$. The set of all feasible plans given an initial value $x_0 \in X$ by

$$\Pi(x_0) = \{\{\tilde{x}_t\}_{t\in\mathbb{N}} \in X^{\mathbb{N}} \mid \tilde{x}_0 = x_0, \quad \tilde{x}_{t+1} \in \Gamma(\tilde{x}_t) \quad \forall t \in \mathbb{N}\}.$$

Defining the function $\mathbb{U} : X^{\mathbb{N}} \to \mathbb{R}$ as

$$\mathbb{U}(\{x_t\}_{t\in\mathbb{N}}) = \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

for any $\{x_t\}_{t\in\mathbb{N}} \in X^{\mathbb{N}}$, we can write the dynamic optimization problem as the unconstrained optimization problem

$$\max_{x\in\Pi(x_0)} \quad \mathbb{U}(x).$$

2.1.1 Existence and Uniquenss of Solutions

Our first order of business is to consider whether there exists a solution to this problem at all. The easiest way to ensure that a solution to a problem exists is by making use of the extreme value theorem; that is, once we show that $\mathbb{U}$ is continuous and $\Pi(x_0)$ is compact with respect to some topology on $X^{\mathbb{N}}$, then the existence of a solution follows from the extreme value theorem.

To this end, we consider the product topology $\tau^{\mathbb{N}}$ on $X^{\mathbb{N}}$, that is, the topology generated by the collection of open rectangles $A_1 \times \cdots \times A_n$, where $A_1, \cdots, A_n \in \tau$, on $X^{\mathbb{N}}$. Our choice of the product topology is convenient because under the axiom of choice, Tychonoff's theorem tells us that $X^{\mathbb{N}}$ is compact under the product topology if (X, τ) is a compact topological space. Therefore, to show that $\Pi(x_0)$ is a compact set under $\tau^{\mathbb{N}}$ when (X, τ) is a compact space, we need only show that $\Pi(x_0)$ is a closed subset of $X^{\mathbb{N}}$. We can show this by induction if (X, τ) is a Hausdorff space and Γ is upper

hemicontinuous and closed-valued:

Lemma (Closedness of Set of Feasible Plans)
Let (X,τ) be a Hausdorff space, and $\Gamma : X \to X$ a non-empty closed valued and upper hemicontinuous correspondence. For any $x_0 \in X$, define

$$\Pi(x_0) = \{\{\tilde{x}_t\}_{t\in\mathbb{N}} \in X^{\mathbb{N}} \mid \tilde{x}_0 = x_0, \quad \tilde{x}_{t+1} \in \Gamma(\tilde{x}_t) \quad \forall t \in \mathbb{N}\}.$$

Then, $\Pi(x_0)$ is a closed subset of $X^{\mathbb{N}}$.

Proof) Fix $x_0 \in X$, and for any $T \in \mathbb{N}$, define

$$\Pi_T(x_0) = \{(x_0, \cdots, x_T) \in X^{T+1} \mid x_{t+1} \in \Gamma(x_t) \quad \forall 0 \le t \le T-1\}.$$

We first show by induction that each $\Pi_T(x_0)$ is a closed subset of X^{T+1} under the product topology τ^{T+1}. Since $\Pi_0(x_0) = \{x_0\}$, and a singleton is closed in Hausdorff space, $\Pi_0(x_0)$ is closed.

Now suppose that for some $T \ge 0$, $\Pi_T(x_0)$ is a closed set in X^{T+1}. Then, we can write

$$\Pi_{T+1}(x_0) = \{(x_0, \cdots, x_T, x_{T+1}) \in X^{T+2} \mid (x_0, \cdots, x_T) \in \Pi_T(x_0), \quad x_{T+1} \in \Gamma(x_T)\}.$$

We show that the complement $\Pi_{T+1}(x_0)^c$ is an open set under τ^{T+2}. Let $(y_0, \cdots, y_T, y_{T+1}) \in \Pi_{T+1}(x_0)^c$. Then, we have two possibilities:

- **Case 1**:
  Suppose that $(y_0, \cdots, y_T) \notin \Pi_T(x_0)$. Then, because $\Pi_T(x_0)$ is a closed subset of X^{T+1} by the inductive hypothesis, there exists a neighborhood U around $(y_0, \cdots, y_T)$ such that $U \subset \Pi_T(x_0)^c$. Therefore,

  $$U \times X \subset \Pi_{T+1}(x_0)^c,$$

  since for any point in $U \times X$, the vector collecting the first $T+1$ entries is not contained in $\Pi_T(x_0)$. Note that $U \times X$ is an open subset of X^{T+2}.

– **Case 2**:

Suppose that $(y_0, \cdots, y_T) \in \Pi_T(x_0)$. Then, it must be the case that $y_{T+1} \notin \Gamma(y_T)$, or $y_{T+1} \in \Gamma(y_T)^c$. Since Γ is a closed-valued correspondence, $\Gamma(y_T)^c$ is an open subset of X. $\{y_{T+1}\}$ is thus a compact subset of X contained in the open set $\Gamma(y_T)^c$; since (X, τ) is a Hausdorff space, there exists an open set $V \in \tau$ with compact closure $\overline{V}$ such that

$$\{y_{T+1}\} \subset V \subset \overline{V} \subset \Gamma(y_T)^c.$$

It follows that $\Gamma(y_T) \subset \overline{V}^c$, where $\overline{V}^c$ is an open set (compact sets are closed in Hausdorff spaces). By the upper hemicontinuity of Γ, there then exists a neighborhood $U \in \tau$ of y_T such that $\Gamma(U) \subset \overline{V}^c$. By implication,

$$V \subset \overline{V} \subset \Gamma(U)^c = \bigcap_{z \in U} \Gamma(z)^c.$$

We can now see that

$$X^T \times U \times V \subset \Pi_{T+1}(x_0),$$

where $(y_0, \cdots, y_T, y_{T+1})$ is contained in $X^T \times U \times V$; this is because, for any $(z_0, \cdots, z_T, z_{T+1}) \in X^T \times U \times V$, since

$$z_{T+1} \in V \subset \Gamma(z_T)^c,$$

we have $z_{T+1} \notin \Gamma(z_T)$. Note once again that $X^T \times U \times V$ is an open subset of X^{T+2}.

In any case, we can find a neighborhood around $(y_0, \cdots, y_T, y_{T+1})$ that is contained within $\Pi_{T+1}(x_0)^c$. It follows that $\Pi_{T+1}(x_0)^c$ is an open and thus $\Pi_{T+1}(x_0)$ a closed subset of X^{T+2} under the product topology τ^{T+2}. By induction, $\Pi_T(x_0)$ is a closed subset of X^{T+1} under τ^{T+1} for any $T \in \mathbb{N}$.

Now note that

$$\Pi(x_0) = \bigcap_{T \in \mathbb{N}} \left(\Pi_T(x_0) \times X^{\mathbb{N}} \right).$$

For any $T \in \mathbb{N}$,

$$\left(\Pi_T(x_0) \times X^{\mathbb{N}}\right)^c = \Pi_T(x_0)^c \times X^{\mathbb{N}};$$

because $\Pi_T(x_0)^c$ is an open subset of X^{T+1}, it follows that $\Pi_T(x_0)^c \times X^{\mathbb{N}}$ is an open rectangle and thus an open subset of $X^{\mathbb{N}}$ under $\tau^{\mathbb{N}}$. Therefore, $\Pi_T(x_0) \times X^{\mathbb{N}}$ is a closed subset of $X^{\mathbb{N}}$ under $\tau^{\mathbb{N}}$, and since the intersection of closed sets is closed, $\Pi(x_0)$ is a closed subset of $X^{\mathbb{N}}$.

Q.E.D.

The other component of the extreme value theorem concerns the continuity of the objective function $\mathbb{U}$. However, this follows easily from an application of Weierstrass' M Test if the components $U : X \times X \to \mathbb{R}$ are bounded and continuous with respect to the product topology τ^2.

Lemma (Continuity of Objective Function)
Let (X, τ) be a topological space, $U : X \times X \to \mathbb{R}$ a bounded function continuous with respect to the product topology τ^2, and $\beta \in (0, 1)$. Then, the function $\mathbb{U} : X^{\mathbb{N}} \to \mathbb{R}$ defined as

$$\mathbb{U}(\{x_t\}_{t \in \mathbb{N}}) = \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

for any $\{x_t\}_{t \in \mathbb{N}} \in X^{\mathbb{N}}$ is continuous with respect to the product topology $\tau^{\mathbb{N}}$.

Proof) Let U be bounded by $M > 0$, that is, $|U(x, y)| \leq M$ for any $(x, y) \in X^2$.

For any $T \in \mathbb{N}_+$, define $\mathbb{U}_T : X^{\mathbb{N}} \to \mathbb{R}$ as

$$\mathbb{U}_T(\{x_t\}_{t \in \mathbb{N}}) = \sum_{t=0}^{T} \beta^t \cdot U(x_t, x_{t+1})$$

for any $\{x_t\}_{t \in \mathbb{N}} \in X^{\mathbb{N}}$. By the continuity of U, the mapping $(x_0, \cdots, x_{T+1}) \mapsto \sum_{t=0}^{T} \beta^t \cdot U(x_t, x_{t+1})$ is a continuous function on X^{T+2}; this means that it is both

upper and lower semicontinuous. For any $a \in \mathbb{R}$, define

$$A_u = \left\{ (x_0, \cdots, x_{T+1}) \in X^{T+2} \mid \sum_{t=0}^{T} \beta^t \cdot U(x_t, x_{t+1}) < a \right\}$$

$$A_l = \left\{ (x_0, \cdots, x_{T+1}) \in X^{T+2} \mid \sum_{t=0}^{T} \beta^t \cdot U(x_t, x_{t+1}) > a \right\};$$

A_u, A_l are both open sets belonging to τ^{T+2}. Then, we can see that

$$\{x \in X^{\mathbb{N}} \mid \mathbb{U}_T(x) < a\} = A_u \times X^{\mathbb{N}}$$
$$\{x \in X^{\mathbb{N}} \mid \mathbb{U}_T(x) > a\} = A_l \times X^{\mathbb{N}},$$

so that $\mathbb{U}_T$ is both upper and lower semicontinuous, and thus continuous on $X^{\mathbb{N}}$. In addition, the boundedness of U implies that $\mathbb{U}_T$ is also bounded; we express this succinctly as $\mathbb{U}_T \in C_b(X^{\mathbb{N}}, \mathbb{R})$, where $C_b(X^{\mathbb{N}}, \mathbb{R})$ is the set of all continuous and bounded real functions on $X^{\mathbb{N}}$.

Now note that, for any $T, S \in \mathbb{N}$ such that $S > T$,

$$|\mathbb{U}_T(x) - \mathbb{U}_S(x)| \leq \sum_{t=T+1}^{S} \beta^t \cdot |U(x_t, x_{t+1})|$$

for any $x = \{x_t\}_{t \in \mathbb{N}} \in X^{\mathbb{N}}$. Since U is bounded, this shows us that

$$\|\mathbb{U}_T - \mathbb{U}_S\|_{\mathcal{C}} = \sup_{x \in X^{\mathbb{N}}} |\mathbb{U}_T(x) - \mathbb{U}_S(x)| \leq M \cdot \left(\sum_{t=T+1}^{S} \beta^t \right),$$

where $\|\cdot\|_{\mathcal{C}}$ is the supremum norm on $C_b(X^{\mathbb{N}}, \mathbb{R})$. Since the series $\sum_t \beta^t$ converges, taking $T, S \to \infty$ on both sides shows us that

$$\lim_{T,S \to \infty} \|\mathbb{U}_T - \mathbb{U}_S\|_{\mathcal{C}} = 0,$$

or that the sequence $\{\mathbb{U}_T\}_{T \in \mathbb{N}}$ is a Cauchy sequence in the supremum norm. Because $(C_b(X^{\mathbb{N}}, \mathbb{R}), \|\cdot\|_{\mathcal{C}})$ is a Banach space, it follows that $\{\mathbb{U}_T\}_{T \in \mathbb{N}}$ converges uniformly to some function in $C_b(X^{\mathbb{N}}, \mathbb{R})$. Finally, since $\mathbb{U}$ is precisely the uniform limit of $\{\mathbb{U}_T\}_{T \in \mathbb{N}}$, we can conclude that $\mathbb{U}$ is a continuous function on $X^{\mathbb{N}}$

with respect to the product topology $\tau^{\mathbb{N}}$.

Q.E.D.

Putting these results together, we are able to formulate the following result on the existence of a solution to the dynamic optimization problem formulated above:

Theorem (Existence of Solution to Dynamic Optimization Problem) Let (X, τ) be a compact Hausdorff space, $\Gamma : X \to X$ a non-empty closed valued and upper hemicontinuous correspondence, $U : X \times X \to \mathbb{R}$ a function continuous with respect to the product topology τ^2, and $\beta \in (0,1)$ some discount factor. Then, there exists a solution to the dynamic optimization problem

$$\max_{\{x_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

$$\text{subject to} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N}$$

for any initial value $x_0 \in X$. If, in addition, $X \subset \mathbb{R}^n$, U is strictly concave and Γ is a convex correspondence, that is,

$$\lambda x' + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)y)$$

for any $\lambda \in [0,1]$ and $x' \in \Gamma(x)$, $y' \in \Gamma(y)$, then this solution is unique.

Proof) Because U is continuous on the compact set $X \times X$, the extreme value theorem tells us that U is bounded. Fixing $x_0 \in X$ and defining $\mathbb{U} : X^{\mathbb{N}} \to \mathbb{R}$ and $\Pi(x_0) \subset X^{\mathbb{N}}$ as above, we can see that, under the stated assumptions, $\Pi(x_0)$ is a compact set and $\mathbb{U}$ is continuous, both with respect to the product topology $\tau^{\mathbb{N}}$. Therefore, by the extreme value theorem, there exists some feasible plan $\{x_t^*\}_{t\in\mathbb{N}} \in \Pi(x_0)$ such that $x_0^* = x_0$ and

$$V(x_0) := \sum_{t=0}^{\infty} \beta^t \cdot U(x_t^*, x_{t+1}^*) = \sup_{\{x_t\}\in\Pi(x_0)} \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1}).$$

Now let U be strictly concave and Γ is a convex correspondence. Suppose there exists a feasible plan $\{\tilde{x}_t\}_{t\in\mathbb{N}} \in \Pi(x_0)$ that also solves the dynamic optimization problem. By definition, $\tilde{x}_0 = x_0$, $\tilde{x}_{t+1} \in \Gamma(\tilde{x}_t)$ for any $t \in \mathbb{N}$, and

$$\sum_{t=0}^{\infty} \beta^t \cdot U(x_t^*, x_{t+1}^*) = \sum_{t=0}^{\infty} \beta^t \cdot U(\tilde{x}_t, \tilde{x}_{t+1}).$$

Putting $\lambda = \frac{1}{2}$, we can then consider the plan $\{x_t\}_{t\in\mathbb{N}}$ defined as $x_t = \lambda\tilde{x}_t + (1-\lambda)x_t^*$ for any $t \in N_+$. Since Γ is convex valued,

$$x_{t+1} = \lambda\tilde{x}_{t+1} + (1-\lambda)x_{t+1}^* \in \Gamma(\lambda\tilde{x}_t + (1-\lambda)x_t^*) = \Gamma(x_t)$$

for any $t \in \mathbb{N}$, so that $\{x_t\}_{t\in\mathbb{N}}$ is a feasible plan. By the definition of maximizers, we must have

$$\sum_{t=0}^{\infty} \beta^t \cdot U(x_t^*, x_{t+1}^*) = \sum_{t=0}^{\infty} \beta^t \cdot U(\tilde{x}_t, \tilde{x}_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1}).$$

The strict concavity of U tells us that

$$\begin{aligned} U(x_t, x_{t+1}) &= U\left(\lambda \cdot (\tilde{x}_t, \tilde{x}_{t+1}) + (1-\lambda) \cdot (x_t^*, x_{t+1}^*)\right) \\ &\geq \lambda \cdot U(\tilde{x}_t, \tilde{x}_{t+1}) + (1-\lambda) \cdot U(x_t^*, x_{t+1}^*), \end{aligned}$$

for any $t \in \mathbb{N}$, where the inequality is an equality if and only if $(\tilde{x}_t, \tilde{x}_{t+1}) = (x_t^*, x_{t+1}^*)$. Therefore, we must have

$$\begin{aligned} 0 &\leq \sum_{t=0}^{\infty} \beta^t \left[U(x_t, x_{t+1}) - \lambda \cdot U(\tilde{x}_t, \tilde{x}_{t+1}) - (1-\lambda) \cdot U(x_t^*, x_{t+1}^*)\right] \\ &= \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1}) - \sum_{t=0}^{\infty} \beta^t \cdot U(x_t^*, x_{t+1}^*) \leq 0, \end{aligned}$$

so that

$$\sum_{t=0}^{\infty} \beta^t \left[U(x_t, x_{t+1}) - \lambda \cdot U(\tilde{x}_t, \tilde{x}_{t+1}) - (1-\lambda) \cdot U(x_t^*, x_{t+1}^*)\right] = 0.$$

Treating the infinite sum as an integral with respect to the counting measure,

the vanishing property for non-negative functions tells us that

$$U(x_t, x_{t+1}) = \lambda \cdot U(\tilde{x}_t, \tilde{x}_{t+1}) + (1 - \lambda) \cdot U(x_t^*, x_{t+1}^*)$$

for any $t \in \mathbb{N}$. By the strict concavity of U. it now follows that $\tilde{x}_t = x_t^*$ for any $t \in \mathbb{N}$, so that $\{x_t^*\}_{t \in \mathbb{N}} \in \Pi(x_0)$ is the unique solution.

Q.E.D.

2.1.2 The Euler Equation and Transversality Condition

Given a dynamic optimization problem of the above form, we can also formulate simple necessary and sufficient conditions that a solution must follow. These are called the euler equation and the transversality condition; we first show that these are sufficient for a feasible plan to solve the dynamic optimzation problem:

Theorem (Sufficient Conditions for Dynamic Optimization)
Let X be a closed convex subset of $\mathbb{R}^n_+$, $\Gamma : X \to X$ a non-empty valued correspondence, $U : X \times X \to \mathbb{R}$ a twice continuously differentiable and concave function, and $\beta \in (0,1)$ some discount factor. Suppose in addition that, for any $x \in X$, $U(\cdot, x)$ is increasing in each of its n entries. For any $(x,y) \in X^2$, denote the gradient of U at (x,y) with respect to the first n entries of U as $U_1(x,y)$, and the gradient with respect to the last n entries as $U_2(x,y)$.

Consider the dynamic optimization problem

$$\max_{\{x_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

$$\text{subject to} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N}$$

for some initial value $x_0 \in \text{int}(X)$. If $\{x_t^*\}_{t\in\mathbb{N}}$ is a feasible plan such that $x_{t+1}^* \in \text{int}(\Gamma(x_t^*))$ and

$$\beta \cdot U_1(x_{t+1}^*, x_{t+2}^*) + U_2(x_t^*, x_{t+1}^*) = \mathbf{0} \qquad \text{(Euler Equation)}$$

$$\lim_{T\to\infty} \beta^T \cdot U_1(x_T^*, x_{T+1}^*)' x_T^* = 0 \qquad \text{(Transversality Condition)}$$

for any $t \in \mathbb{N}$, then $\{x_t^*\}_{t\in\mathbb{N}}$ is a solution to the above problem.

Proof) Consider any feasible plan $\{x_t\}_{t\in\mathbb{N}} \in \Pi(x_0)$. For any $t \in \mathbb{N}$, consider the second degree Taylor expansion of $U(x_t, x_{t+1})$ around (x_t^*, x_{t+1}^*):

$$U(x_t, x_{t+1}) = U(x_t^*, x_{t+1}^*) + \nabla U(x_t^*, x_{t+1}^*)' \begin{pmatrix} x_t - x_t^* \\ x_{t+1} - x_{t+1}^* \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} x_t' - x_t^{*\prime} & x_{t+1}' - x_{t+1}^{*\prime} \end{pmatrix} D^2U(\tilde{x}_t, \tilde{x}_{t+1}) \begin{pmatrix} x_t - x_t^* \\ x_{t+1} - x_{t+1}^* \end{pmatrix},$$

where $(\tilde{x}_t, \tilde{x}_{t+1}) \in X^2$ is some convex combination of (x_t^*, x_{t+1}^*) and (x_t, x_{t+1}), and $D^2U(\tilde{x}_t, \tilde{x}_{t+1})$ is the Hessian matrix evaluated at $(\tilde{x}_t, \tilde{x}_{t+1})$. Since U is concave on X^2, it follows that $D^2U(\tilde{x}_t, \tilde{x}_{t+1})$ is negative semidefinite and therefore

$$\begin{aligned} U(x_t, x_{t+1}) &\leq U(x_t^*, x_{t+1}^*) + \boldsymbol{\nabla} U(x_t^*, x_{t+1}^*)' \begin{pmatrix} x_t - x_t^* \\ x_{t+1} - x_{t+1}^* \end{pmatrix} \\ &= U(x_t^*, x_{t+1}^*) + U_1(x_t^*, x_{t+1}^*)'(x_t - x_t^*) + U_2(x_t^*, x_{t+1}^*)'(x_{t+1} - x_{t+1}^*). \end{aligned}$$

This holds for any $t \in \mathbb{N}$, so for any $T \in N_+$,

$$\begin{aligned} \Delta_T &:= \sum_{t=0}^{T} \beta^t (U(x_t^*, x_{t+1}^*) - U(x_t, x_{t+1})) \\ &\geq \sum_{t=0}^{T} \beta^t \cdot U_1(x_t^*, x_{t+1}^*)'(x_t^* - x_t) + \sum_{t=0}^{T} \beta^t \cdot U_2(x_t^*, x_{t+1}^*)'(x_{t+1}^* - x_{t+1}) \\ &= \sum_{t=0}^{T} \beta^t \left[\beta \cdot U_1(x_{t+1}^*, x_{t+2}^*) + U_2(x_t^*, x_{t+1}^*)\right]' (x_{t+1}^* - x_{t+1}) \\ &\quad - \beta^{T+1} \cdot U_1(x_{T+1}^*, x_{T+2}^*)'(x_{T+1}^* - x_{T+1}). \end{aligned}$$

By the Euler equation,

$$\beta \cdot U_1(x_{t+1}^*, x_{t+2}^*) + U_2(x_t^*, x_{t+1}^*) = \mathbf{0}$$

for any $t \in \mathbb{N}$, so we have

$$\Delta_T \geq -\beta^{T+1} \cdot U_1(x_{T+1}^*, x_{T+2}^*)'(x_{T+1}^* - x_{T+1}).$$

for any $T \in N_+$. Since $U(\cdot, x)$ is increasing for each of its n entries for any value of $x \in X$, it must be the case that $U_1(x_{T+1}^*, x_{T+2}^*) \geq \mathbf{0}$. In addition, $x_{T+1} \in X \subset \mathbb{R}_+^n$, so we can see that

$$U_1(x_{T+1}^*, x_{T+2}^*)' x_{T+1} \geq 0.$$

Therefore,

$$\Delta_T \geq -\beta^{T+1} \cdot U_1(x^*_{T+1}, x^*_{T+2})' x^*_{T+1}.$$

Taking $T \to \infty$ implies, by the transversality condition, that

$$\lim_{T\to\infty} \Delta_T \geq 0.$$

Therefore,

$$0 \leq \lim_{T\to\infty} \Delta_T = \sum_{t=0}^{\infty} \beta^t (U(x^*_t, x^*_{t+1}) - U(x_t, x_{t+1})) = \mathbb{U}(\{x^*_t\}_{t\in\mathbb{N}}) - \mathbb{U}(\{x_t\}_{t\in\mathbb{N}}),$$

so that

$$\mathbb{U}(\{x^*_t\}_{t\in\mathbb{N}}) \geq \mathbb{U}(\{x_t\}_{t\in\mathbb{N}}).$$

This holds for any feasible plan $\{x_t\}_{t\in\mathbb{N}} \in \Pi(x_0)$, so it follows that $\{x^*_t\}_{t\in\mathbb{N}}$ is a solution to the dynamic optimization problem under the initial value $x_0 \in X$.

Q.E.D.

2.2 The Neoclassical Growth Model in Discrete Time

Here we study the neoclassical growth model as an application of the deterministic dynamic model presented in the preceding chapter.

2.2.1 Model Assumptions

The basic building blocks of the elementary neoclassical growth model are given as follows:

Commodities There is one consumption good and two inputs, labor and capital. The consumption good and capital are non-negative reals, and labor takes values in the interval $[0, 1]$.

Preferences There is a single (representative) consumer in the economy who also acts as the representative firm. Their instantaenous utility at time t depends on the amount of goods they consume, c_t, and the amount of leisure that they enjoy, $1 - h_t$, where h_t is the labor hours supplied. Their intertemporal utility given the streams $\{c_t\}_{t\in\mathbb{N}}$ and $\{h_t\}_{t\in\mathbb{N}}$ of consumption and labor is intertemporally separable and is given as

$$W(\{c_t, h_t\}_{t\in\mathbb{N}}) = \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, 1 - h_t)$$

for some discount factor $\beta \in (0, 1)$, where the instantaneous utility function $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is bounded and twice continuously differentiable. In addition,

$$u_c(c, l) > 0, \quad u_l(c, l) > 0$$

for any $(c, l) \in \mathbb{R}_+ \times [0, 1]$, where the subscripts represent derivatives with respect to the associated variables. We also assume that u is strictly concave.

Technologies The consumption good each period is produced using the inputs of capital and labor hours. There exists a production function $F : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ such that

$$y_t = F(k_t, h_t)$$

for any $t \in \mathbb{N}$, where y_t is the amount of consumption goods produced at time t. The household then either consumes y_t or invests it:

$$c_t + i_t = y_t.$$

Finally, investment determines the amount of capital in the next period, where $\delta \in (0,1)$ is the depreciation rate of capital:

$$k_{t+1} = i_t + (1-\delta)k_t.$$

We require there to be a non-negative amount of capital each period, so that $k_{t+1} \geq 0$ for any $t \in \mathbb{N}$ and the initial amount of capital is given as $k_0 \geq 0$. The relationship between production, consumption and investment at time t can be summarized as

$$c_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t, \quad k_{t+1} \geq 0, c_t \geq 0,$$

or equivalently,

$$k_{t+1} \in [0,\ F(k_t, h_t) + (1-\delta)k_t], \quad c_t = F(k_t, h_t) + (1-\delta)k_t - k_{t+1}.$$

We assume that the production function F is twice continuously differentiable, with the properties

$$\begin{aligned} F_k(k,h) > 0, \quad & F_h(k,h) > 0, \\ F_{kk}(k,h) < 0 \quad & F_{hh}(k,h) < 0 \end{aligned}$$

for any $(k,h) \in \mathbb{R}_+ \times [0,1]$, so that the marginal product for both inputs is positive and diminishing. We also require F to be concave (perhaps not strictly so). This encompasses the case where F is CRS.

Note that, for any $(k,h) \in \mathbb{R}_+ \times [0,1]$, $F_k(k,h) > 0$ nad $F(0,h) = 0$ implies that the mapping $k \mapsto F(k,h) + (1-\delta)k$ is always positive for positive k and strictly increasing in k, since $F(0,h) + (1-\delta) \cdot 0 = 0$ and

$$\frac{\partial}{\partial k}\left(F(k,h) + (1-\delta)k\right) = F_k(k,h) + 1 - \delta > 0$$

for any $(k,h) \in \mathbb{R}_+ \times [0,1]$.

Finally, we assume that the the following Inada conditions hold:

$$\lim_{k \downarrow 0} F_k(k,h) = +\infty$$
$$\lim_{k \to \infty} F_k(k,h) = 0$$

for any $h > 0$.

The representative household's problem is then equivalent to the social planner's problem under the feasibility constraint:

$$\begin{aligned} \max_{\{c_t,h_t\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, 1-h_t) \\ \text{subject to} \quad & c_t + k_{t+1} = F(k_t,h_t) + (1-\delta)k_t, \\ & k_{t+1} \geq 0,\ c_t \geq 0 \quad \text{for any } t \in \mathbb{N} \end{aligned}$$

given the initial capital stock $k_0 > 0$. To rewrite this problem in terms of the dynamic optimization problem we studied in the preceding section, we first define $x_t = (k_t, h_{t-1})$ for any $t \in \mathbb{N}$, $X = \mathbb{R}_+ \times [0,1]$, and the correspondence $\Gamma : X \to X$ as

$$\Gamma(k,h) = \{(k',h') \in X \mid 0 \leq k' \leq F(k,h') + (1-\delta)k\}$$

for any $(k,h) \in X$. Furthermore, we define the function $U : X \times X \to \mathbb{R}$ as

$$U((k,h),(k',h')) = u(F(k,h') + (1-\delta)k - k',\ 1-h')$$

for any $(k,h),(k',h') \in X$. Then, the social planner's problem can be written as

$$\max_{\{x_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

$$\text{subject to} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N}$$

given intial values $x_0 = (k_0, 1) \in X$.

2.2.2 Existence and Uniqueness of Solution

We want to know whether this problem has a solution. To this end, we return to the existence theorem. For the problem above to admit a solution, a sufficient condition is that U is continuous on $X \times X$, Γ is non-empty closed valued and upper hemicontinuous, and X is compact. We immediately run into a problem with X, since $\mathbb{R}_+$ is not compact. However, we can show that, for any feasible plan $\{x_t\}_{t\in\mathbb{N}}$ under an appropriate initial value $x_0 \in X$, there must exist some $\hat{k} > 0$ such that

$$x_t \in [0, \hat{k}] \times [0, 1]$$

for any $t \in \mathbb{N}$. In other words, we can restrict our focus to the compact set $[0,\hat{k}] \times [0,1]$:

Lemma (Maximum Maintainable and Attainable Capital Stock)
Suppose the assumptions above hold. Then, there exists a unique $\hat{k} > 0$ and some $M > 0$ such that, for any initial value $k_0 \leq \hat{k} + M$, any feasible plan $\{(k_{t+1}, h_t)\}_{t\in\mathbb{N}}$ starting from k_0 satisfies, for any $t \in \mathbb{N}$, the following conditions:

i) If $k_t > \hat{k}$ then $k_{t+1} < k_t$.

ii) $k_t \in [0, \hat{k} + M]$.

Proof) Define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ as $f(k) = F(k,1)$ for any $k \geq 0$. Then, $f(0) =$

$F(0,1) = 0$,

$$f'(k) = F_k(k,1) > 0 \quad \text{and} \quad f''(k) = F_{kk}(k,1) < 0$$

for any $k > 0$, so that f is a strictly concave function on $[0,+\infty)$. Now define $g : \mathbb{R}_+ \to \mathbb{R}$ as

$$g(k) = f(k) - \delta k$$

for any $k \geq 0$. g is then a twice continuously differentiable function with

$$g'(k) = f'(k) - \delta, \quad g''(k) = f''(k) < 0$$

for any $k > 0$, so that g' is strictly decreasing on $(0,+\infty)$ and g is a strictly concave function.

The Inada conditions can be written in terms of f as

$$\lim_{k\to\infty} f'(k) = 0, \quad \lim_{k\downarrow 0} f'(k) = +\infty.$$

As such, there exists an interval $(a,b) \subset \mathbb{R}_{++}$ such that $f'(b) < \delta < f'(a)$; by the intermediate value theorem, there exists some $k^* > 0$ such that $f'(k^*) = \delta$, or $g'(k*) = 0$. By the strict concavity of g, it follows that k^* is the unique solution to the problem

$$\max_{k\in(0,+\infty)} \quad g(k).$$

Furthermore, because g' is strictly decreasing, $g'(k) > 0$ for any $k < k^*$ and $g'(k) < 0$ for any $k > k^*$. Define $M = g(k^*)$.

We now proceed in steps:

- **g is positive for any k small enough**
  By the Inada conditions, $\lim_{k\downarrow 0} g'(k) = +\infty$ and the derivative g' is strictly decreasing on $(0,+\infty)$, so there exists some $k_1 > 0$ such that $g'(k) = f'(k) - \delta > 0$ for any $0 < k \leq k_1$. By the mean value theorem, this indicates that $g(k) > 0$ for any $0 < k \leq k_1$. By implication, $M > 0$, since $M = g(k^*) > g(k_1) > 0$.

– **g is negative for any k large enough**
By the Inada conditions again, $\lim_{k\to\infty} g'(k) = -\delta$. Thus, there exists some $\tilde{k} > 0$ such that

$$-\delta \leq g'(k) \leq g'(\tilde{k}) < 0$$

for any $\tilde{k} \leq k$. Since g' is strictly decreasing, that $g'(\tilde{k}) < 0 < g'(k_1)$ indicates that $k_1 < \tilde{k}$.

If $g(\tilde{k}) < 0$, we are done, since $g(k) < g(\tilde{k}) < 0$ for any $k > \tilde{k}$ because g' is negative on $[\tilde{k}, +\infty)$. Put $k_2 = \tilde{k}$ in this case.

On the other hand, suppose that $g(\tilde{k}) \geq 0$. Choose any $\epsilon > 0$ so that

$$\epsilon > -\frac{g(\tilde{k})}{g'(\tilde{k})} \geq 0.$$

Then, $g'(\tilde{k})\epsilon < -g(\tilde{k})$ and, by the mean value theorem, there exists a $t \in [0,1]$ such that

$$g(\tilde{k}+\epsilon) - g(\tilde{k}) = g'(\tilde{k}+t\epsilon)\epsilon.$$

Since $g'(\tilde{k}+t\epsilon) < g'(\tilde{k})$, this implies that

$$g(\tilde{k}+\epsilon) - g(\tilde{k}) < g'(\tilde{k})\epsilon < -g(\tilde{k}),$$

and thus $g(\tilde{k}+\epsilon) < 0$. Define $k_2 = \tilde{k}+\epsilon > k_1$. For any $k > k_2$, since $g'(k) < 0$ for any $k \geq k_2 \geq \tilde{k}$, the mean value theorem tells us that $g(k) < 0$.

– **g has a unique zero on $(0, +\infty)$**
g is a continuous function such that $g(k_1) > 0 > g(k_2)$. Therefore, by the intermediate value theorem there exists a $\hat{k} \in (k_1, k_2)$ such that $g(\hat{k}) = 0$.

We now show that this $\hat{k}$ is unique on $(0, +\infty)$.

First, note that $\hat{k} > k^*$. If $\hat{k} = k^*$, then $g(\hat{k}) = 0$, a contradiction. On the other hand, if $\hat{k} < k^*$, then by the mean value theorem,

$$0 = g(\hat{k}) - g(0) = g'(k) \cdot \hat{k},$$

or $g'(k) = 0$, for some $k \in (0, \hat{k}) \subset (0, k^*)$. This contradicts the fact that $g' > 0$ for any $k \in (0, k^*)$, so it must be the case that $\hat{k} > k^*$. By implication, $g'(\hat{k}) < 0$.

Suppose $g(k') = 0$ for some $k' > 0$ such that $k' \neq \hat{k}$. By the same line of reasoning as above, we must have $k' > k^*$. By the mean value theorem, however, there must exist some $\lambda \in (0, 1)$ such that

$$0 = g(k') - g(\hat{k}) = g'(\lambda k' + (1 - \lambda)\hat{k})(k' - \hat{k}),$$

or $g'(\lambda k' + (1 - \lambda)\hat{k}) = 0$. $k', \hat{k}$ are both larger than k^*, so it must be the case that $\lambda k' + (1 - \lambda)\hat{k} > k^*$ as well. This contradicts the fact that k^* is the unique solution to $g' = 0$. Therefore, $\hat{k}$ is the unique solution to the equation $g = 0$.

– $g(k) \geq 0$ **if and only if** $k \in [0, \hat{k}]$

Suppose $g(k) < 0$ for some $0 < k < \hat{k}$. This k would have to be larger than k_1, since g is greater than 0 for any positive values smaller than or equal to k_1. By the mean value theorem, then, there exists a point $k_1 < k_1^* < k$ such that

$$g'(k_1^*) = \frac{g(k) - g(k_1)}{k - k_1} < 0.$$

On the other hand, by the mean value theorem again, there exists a point $k < k_2^* < \hat{k}$ such that

$$g'(k_2^*) = \frac{g(\hat{k}) - g(k)}{\hat{k} - k} > 0.$$

However, this contradicts the fact that g' is a decreasing function on $\mathbb{R}_{++}$, so we must have $g(k) \geq 0$.

Conversely, suppose $g(k) \geq 0$ for some $\hat{k} < k$. We must have $g(k) > 0$ by the uniqueness of $\hat{k}$ as a non-zero solution of $g = 0$. In this case, there exists a $\hat{k} < k^* < k$ such that

$$g'(k^*) = \frac{g(k) - g(\hat{k})}{k - \hat{k}} > 0.$$

Since g' is a strictly decreasing function, this means that $g' > 0$ on $(0, k^*]$, which implies that $g(\hat{k}) > g(k_1) > 0$, a contradiction. Therefore, we must have $g(k) < 0$ for any $\hat{k} < k$.

So far, we have established that:

- **Properties of Maximizer of g**
  There exists a unique $k^* > 0$ such that $g'(k^*) = 0$ and

  $$M := g(k^*) = \max_{k>0} g(k) > 0.$$

  Furthermore, $g'(k_1) > 0 > g'(k_2)$ for any $0 < k_1 < k^* < k_2$.

- **Properties of Zero of g**
  There exists a unique $\hat{k} > k^*$ such that $g(\hat{k}) = 0$. Furthermore, $g(k_1) > 0 > g(k_2)$ for any $0 < k_1 < \hat{k} < k_2$.

Choose an initial level of capital $k_0 \leq \hat{k} + M$. Then, we can inductively show that any feasible plan $\{k_{t+1}, h_t\}_{t \in \mathbb{N}}$ must satisfy $0 \leq k_{t+1} \leq \hat{k} + M$ for any $t \in \mathbb{N}$. Since the case for $t = 0$ is trivially satisfied, suppose that $k_t \leq \hat{k} + M$ for some $t \in \mathbb{N}$. We consider the following cases.

1. $k_t < \hat{k}$

Note that

$$\begin{aligned} 0 \leq k_{t+1} &\leq F(k_t, h_t) + (1-\delta)k_t \\ &\leq F(k_t, 1) + (1-\delta)k_t \qquad (F_h \text{ is positive on } X) \\ &= g(k_t) + k_t \leq M + \hat{k}, \end{aligned}$$

where the last inequality follows because M is the maximum value of g.

2. $k_t \geq \hat{k}$
   In this case, $g(k_t) \leq 0$, so that

$$0 \leq k_{t+1} \leq g(k_t) + k_t \leq k_t \leq \hat{k} + M,$$

   where the last inequality follows from the inductive hypothesis.

Therefore, $k_{t+1} \in [0, \hat{k} + M]$, and by induction,

$$\{k_t\}_{t \in \mathbb{N}} \subset [0, \hat{k} + M]^{\mathbb{N}}.$$

Q.E.D.

From the theorem above, we call $\hat{k}$ the maximum maintainable capital stock, and $\hat{k} + M$ the maximum attainable capital stock. Restricting our attention to the set of feasible capital stock bounded above by the maximum attainable capital stock, the existence of a unique solution now follows easily:

Theorem (Existence and Uniqueness of Solution to Growth Model) Suppose the assumptions above hold, and let $\hat{k} > 0$ be the maximum maintainable capital stock. Then, there exists a feasible plan $\{c_t^*, h_t^*\}_{t\in\mathbb{N}}$ that uniquely solves the problem

$$\begin{aligned} \max_{\{c_t,h_t\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, 1-h_t) \\ \text{subject to} \quad & c_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t, \\ & k_{t+1} \geq 0,\ c_t \geq 0 \quad \text{for any } t \in \mathbb{N} \end{aligned}$$

given the initial capital stock $0 < k_0 \leq \hat{k}$.

Proof) In light of the preceding lemma, this problem is equivalent to solving the dynamic optimization problem

$$\begin{aligned} \max_{\{x_t\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1}) \\ \text{subject to} \quad & x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N} \end{aligned}$$

given the initial value $x_0 = (k_0, 1) \in X$, where:

- $x_t = (k_t, h_{t-1})$ for any $t \in \mathbb{N}$,
- $X = [0, \hat{k}] \times [0, 1]$,
- $\Gamma : X \to X$ is the correspondence defined as

$$\Gamma(x) = \{y \in X \mid 0 \leq y_1 \leq F(x_1, y_2) + (1-\delta)x_1\}$$

  for any $x = (x_1, x_2) \in X$, and
- $U : X \times X \to \mathbb{R}$ is the function defined as

$$U(x, y) = u(F(x_1, y_2) + (1-\delta)x_1 - y_1,\ 1 - y_2)$$

  for any $x, y \in X$.

Here, X is a compact set, and U is a bounded and continuous function because both u and F are continuous functions, and u is also bounded. Furthermore, for

any $\lambda \in (0,1)$ and distinct $(x,y),(x',y') \in X \times X$, we have

$$
\begin{aligned}
&U\left[\lambda(x,y)+(1-\lambda)(x',y')\right] = U\left(\lambda x+(1-\lambda)x',\ \lambda y+(1-\lambda)y'\right) \\
&= u\left[F(\lambda\cdot(x_1,y_2)+(1-\lambda)(x_1',y_2'))+(1-\delta)(\lambda x_1+(1-\lambda)x_1')-\left(\lambda\cdot y_1+(1-\lambda)y_1'\right),\ 1-(\lambda y_2+(1-\lambda)y_2')\right] \\
&\geq u\left[\lambda\cdot F(x_1,y_2)+(1-\lambda)F(x_1',y_2')+(1-\delta)(\lambda x_1+(1-\lambda)x_1')-\left(\lambda\cdot y_1+(1-\lambda)y_1'\right),\ 1-(\lambda y_2+(1-\lambda)y_2')\right] \\
&> \lambda\cdot u[F(x_1,y_2)+(1-\delta)x_1-y_1,\ 1-y_2]+(1-\lambda)\cdot u\left[F(x_1',y_2')+(1-\delta)x_1'-y_1',\ 1-y_2'\right] \\
&= \lambda\cdot U(x,y)+(1-\lambda)\cdot U(x',y'),
\end{aligned}
$$

so that U is strictly concave on $X \times X$.

As for Γ, it is non-empty valued because

$$F(x_1,y_2)+(1-\delta)x_1 \geq 0$$

for any $x_1 \geq 0$ and $y_2 \in [0,1]$. In addition, it is closed valued; to see this, fix $x \in X$ and let $\{y^{(n)}\}_{n\in N_+}$ be a sequence converging to some $y \in \Gamma(x)$ such that $y^{(n)} \in \Gamma(x)$ for any $n \in N_+$. Then, $y \in X$ because X is closed, and

$$0 \leq y_1^{(n)} \leq F(x_1,y_2^{(n)})+(1-\delta)x_1$$

for any $n \in N_+$. Taking $n \to \infty$ implies, by the continuity of F, that

$$0 \leq y_1 \leq F(x_1,y_2)+(1-\delta)x_1,$$

so that $y \in \Gamma(x)$.

It remains to show that Γ is upper hemicontinuous. For this, we rely on the characterization of uhc for non-empty and closed valued correspondences whose target space X is compact. Choose any $x \in X$, a sequence $\{x^{(n)}\}_{n\in N_+}$ converging to x, and a convergent sequence $\{y^{(n)}\}_{n\in N_+}$ with limit $y \in X$ such that $y^{(n)} \in \Gamma(x^{(n)})$ for any $n \in N_+$. By definition, for any $n \in N_+$,

$$0 \leq y_1^{(n)} \leq F(x_1^{(n)},y_2^{(n)})+(1-\delta)x_1^{(n)},$$

so taking $n \to \infty$ on both sides shows us that

$$0 \leq y_1 \leq F(x_1, y_2) + (1-\delta)x_1.$$

Therefore, $y \in \Gamma(x)$ and Γ is upper hemicontinuous at x. This holds for any $x \in X$, so Γ is uhc.

Finally, for any $\lambda \in [0,1]$, $x, x' \in X$ and y, y' such that $y \in \Gamma(x)$ and $y' \in \Gamma(x')$, since

$$0 \leq y_1 \leq F(x_1, y_2) + (1-\delta)x_1.$$

and

$$0 \leq y_1' \leq F(x_1', y_2') + (1-\delta)x_1',$$

the concavity of F tells us that

$$\begin{aligned} 0 \leq \lambda y_1 + (1-\lambda)y_1' &\leq \lambda \cdot F(x_1, y_2) + (1-\lambda)F(x_1', y_2') + (1-\delta)(\lambda x_1 + (1-\lambda)x_1') \\ &\leq F\left[\lambda \cdot (x_1, x_1') + (1-\lambda) \cdot (y_2, y_2')\right] + (1-\delta)(\lambda x_1 + (1-\lambda)x_1'). \end{aligned}$$

Therefore, $\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda_2)x')$, making Γ a convex (be careful not to confuse this with convex valued) correspondence.

The conditions for the existence theorem in the previous section are now satisfied, so from that theorem we can conclude that there exists a unique solution $\{k_t^*, h_{t-1}^*\}_{t \in \mathbb{N}}$ to the above dynamic optimization problem. Defining

$$c_t^* = F(k_t^*, h_t^*) + (1-\delta)h_t^* - k_{t+1}^*$$

for any $t \in \mathbb{N}$, it follows that $\{c_t^*, h_t^*\}_{t \in \mathbb{N}}$ is the unique solution to the original problem.

Q.E.D.

2.2.3 Dynamics of the Optimal Path

Now that we have established the existence of a unique optimal consumption/labor path $\{c_t^*, h_t^*\}_{t\in\mathbb{N}}$ for the neoclassical growth model, we can use the Euler equation and transversality condition to characterize this optimal path. Consider the reparametrized version of the social planner's problem that we considered in the previous theorem:

$$\begin{aligned} \max_{\{x_t\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1}) \\ \text{subject to} \quad & x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \in \mathbb{N}, \end{aligned}$$

given intial value $0 < k_0 \leq \hat{k}$. The euler equation is given as

$$\beta \cdot U_1(x_{t+1}^*, x_{t+2}^*) + U_2(x_t^*, x_{t+1}^*) = \mathbf{0},$$

where

$$\begin{aligned} U_1(x_{t+1}^*, x_{t+2}^*) &= \begin{pmatrix} u_c(F(k_{t+1}^*, h_{t+1}^*) + (1-\delta)k_{t+1}^* - k_{t+2}^*,\ 1 - h_{t+1}^*) \cdot \Big(F_k(k_{t+1}^*, h_{t+1}^*) + 1 - \delta\Big) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} u_c(c_{t+1}^*, 1 - h_{t+1}^*) \cdot \Big(F_k(k_{t+1}^*, h_{t+1}^*) + 1 - \delta\Big) \\ 0 \end{pmatrix} \end{aligned}$$

and

$$U_2(x_t^*, x_{t+1}^*) = \begin{pmatrix} -u_c(c_t^*, 1 - h_t^*) \\ u_c(c_t^*, 1 - h_t^*) F_h(k_t^*, h_t^*) - u_l(c_t^*, 1 - h_t^*) \end{pmatrix}.$$

Therefore, we obtain two Euler equations:

$$u_c(c_t^*, 1 - h_t^*) = \beta \cdot u_c(c_{t+1}^*, 1 - h_{t+1}^*) \cdot \Big(F_k(k_{t+1}^*, h_{t+1}^*) + 1 - \delta\Big)$$

(Intertemporal Substitution of Consumption)

$$u_l(c_t^*, 1 - h_t^*) = u_c(c_t^*, 1 - h_t^*) \cdot F_h(k_t^*, h_t^*)$$

(Intratemporal Substitution of Consumption and Labor)

The transversality condition becomes

$$\lim_{T\to\infty} \beta^T u_c(c_T^*, 1-h_T^*)\cdot\left(F_k(k_T^*, h_T^*)+1-\delta\right)k_T^* = 0,$$

which, in light of the Euler equations, becomes

$$\lim_{T\to\infty} \beta^T u_c(c_T^*, 1-h_T^*)k_{T+1}^* = 0.$$

We now interpret these equations. Under a perfectly competitive factor market, the marginal products of capital and labor satisfy:

$$F_h(k_t^*, h_t^*) = w_t \quad \text{and} \quad F_k(k_t^*, h_t^*)+1-\delta = 1+r_t,$$

where w_t and r_t are the real wage and real interest rate at time t. Therefore, the Euler equations can now be written as

$$\begin{aligned} MU_{c,t} &= \beta\cdot MU_{c,t+1}\cdot(1+r_t) \\ MU_{l,t} &= MU_{c,t}\cdot w_t, \end{aligned}$$

where MU_c and MU_l are the marginal utilites of leisure and consumption. The second equation tells us that, under separable consumption and labor (the marginal utility of consumption does not depend on leisure, and vice versa), an increase in the wage leads to an increase in the marginal utility of leisure, which leads to higher labor supply. Thus, the intratemporal euler equation implies an upward sloping labor supply curve.

2.3 Dynamic Optimization Problems in Continuous Time

Models in continuous time often prove more useful to study the dynamics of macroeconomic models. For this reason, we present here the continuous time counterpart of the discrete time dynamic optimization problem that we studied in the previous sections. The mathematics of continuous time problems are more invovled than those of problems in discrete time. For instance, there are more stringent conditions required to ensure that a solution to a dynamic optimization problem is unique under continuous time, and proving the existence of a solution is made difficult when the time index is not compact, as in the case of infinite horizon problems. As such, we slightly relax the mathematical formalism in this section, and focus on the implications of the methodology in the context of the neoclassical growth model, also known as the Ramsey-Cass-Koopmans (RCK) model.

We consider problems of the following type. Let $\mathbb{T}$ be the time index; we usually take it to be a finite interval $[0,T]$ or $[0,+\infty)$ in the case of infinite horizonn problems. For simplicity, we consider problems with univariate states and multivariate controls. The set of all possible controls $\mathcal{U}$ is a subset of the set $C(\mathbb{T},\mathbb{R}^n)$ of continuous functions from $\mathbb{T}$ to $\mathbb{R}^n$ that take values in the set $U \subset \mathbb{R}^n$. Likewise, the set of all possible states $\mathcal{X}$ is taken to be a subset of $C(\mathbb{T},\mathbb{R}^m)$, or continuous functions from $\mathbb{T}$ to $\mathbb{R}^m$, that take values in some set $X \subset \mathbb{R}^m$.

The felicity function, or instantaneous objective function, is denoted $h : U \times X \times \mathbb{T} \to \mathbb{R}_+$; h is a function of the control, state and time. We choose h so that

$$\int_{\mathbb{T}} h(u(t),x(t),t)dt < +\infty$$

for any $(u,x) \in \mathcal{U} \times \mathcal{X}$; if $\mathbb{T} = [0,T]$, it suffices by the extreme value theorem to require the continuity of h. If $\mathbb{T} = [0,+\infty)$, we usually assume that h assumes the discounted form

$$h(u,x,t) = e^{-\rho t}F(u,x)$$

for any $(u,x,t) \in U \times X \times \mathbb{T}$, where F is bounded and continuous and $\rho > 0$. In this case,

letting $M > 0$ be the bound of F,

$$\int_0^\infty h(u(t), x(t), t)dt \leq M \cdot \int_0^\infty e^{-\rho t}dt = M \cdot \rho < +\infty$$

for any $(u, x) \in \mathcal{U} \times \mathcal{X}$. Under these conditions ensuring the finiteness of the integral for any control-state combination, we define the objective function $J : \mathcal{U} \times \mathcal{X} \to \mathbb{R}_+$ as

$$J(u, x) = \int_0^\infty h(u(t), x(t), t)dt$$

for any $(u, x) \in \mathcal{U} \times \mathcal{X}$.

Given a path $u \in \mathcal{U}$ of the control variable and an initial value x_0 of the state variable, the associated state variable $x \in \mathcal{X}$ evolves according to the differential equation

$$\dot{x}(t) = g(u(t), x(t), t),$$

and initial value $x(0) = x_0$, where $g : U \times X \times \mathbb{T} \to \mathbb{R}^m$. Our dynamic optimization problem can now be posed as follows:

$$\max_{u \in \mathcal{U}} \quad \int_0^\infty h(u(t), x(t), t)dt$$

$$\text{subject to} \quad x(0) = x_0, \quad \dot{x}(t) = g(u(t), x(t), t) \quad \text{for any } t \geq 0.$$

Instead of dealing with existence and uniqueness, we turn directly to the counterpart of the Euler equations and transversality condition, namely the Pontryagin conditions on the Hamiltonian. Given the above dynamic optimization problem, the corresponding Hamiltonian is the function $H : U \times X \times \mathbb{R}^m_+ \times [0, +\infty) \to \mathbb{R}$ defined as

$$H(u, x, \lambda, t) = h(u, x, t) + \lambda' g(u, x, t)$$

for any $(u, x, \lambda, t) \in U \times X \times \mathbb{R}^m \times [0, +\infty)$. The theorem below provides sufficient conditions for a maximizer of the dynamic optimization problem above:

Theorem (Pontryagin Maximum Principle)
Assume $X \subset \mathbb{R}^m_+$, and consider the dynamic optimization problem

$$\begin{aligned} \max_{u \in \mathcal{U}} \quad & \int_0^\infty h(u(t), x(t), t)dt \\ \text{subject to} \quad & x(0) = x_0, \\ & \dot{x}(t) = g(u(t), x(t), t) \quad \text{for any } t \geq 0, \end{aligned}$$

where we assume that h and the components $g_1, \cdots, g_m$ of g are functions that are concave and twice continuously differentiable in their first two arguments. Let $H : U \times X \times \mathbb{R}^m \times [0, +\infty) \to \mathbb{R}$ be the Hamiltonian corresponding to this problem, and let $\hat{u} \in \mathcal{U}$, $\hat{x} \in \mathcal{X}$, and $\hat{\lambda} \in C([0, +\infty), \mathbb{R}^m_+)$ functions that satisfy the following conditions:

$$\frac{\partial H}{\partial u} = \boldsymbol{\nabla}_u h(\hat{u}(t), \hat{x}(t), t) + D_u g(\hat{u}(t), \hat{x}(t), t)' \, \hat{\lambda}(t) = \mathbf{0} \qquad \text{(Control Equation)}$$

$$\frac{\partial H}{\partial x} = \boldsymbol{\nabla}_x h(\hat{u}(t), \hat{x}(t), t) + D_x g(\hat{u}(t), \hat{x}(t), t)' \, \hat{\lambda}(t) = -\dot{\hat{\lambda}}(t) \qquad \text{(State Equation)}$$

$$\frac{\partial H}{\partial \lambda} = g(\hat{u}(t), \hat{x}(t), t) = \dot{\hat{x}}(t) \qquad \text{(Costate Equation)}$$

$$\lim_{T \to \infty} \hat{\lambda}(T)' \, \hat{x}(T) = 0 \qquad \text{(Transversality Condition)}$$

for any $t \geq 0$, where

$$D_u g = \begin{pmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_n} \end{pmatrix} \in \mathbb{R}^{m \times n} \quad \text{and} \quad D_x g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{pmatrix} \in \mathbb{R}^{m \times m}$$

are the Jacobians of g with respect to u and x, respectively. Then, $\hat{u} \in \mathcal{U}$ is a solution to the dynamic optimization problem above.

Proof) Let $(u, x) \in \mathcal{U} \times \mathcal{X}$ be another pair of continuous functions that satify the constraints of the optimization problem: that is, $x(0) = x_0$ and

$$\dot{x}_t = g(u(t), x(t), t)$$

for any $t \geq 0$. We will show that $J(\hat{u}, \hat{x}) \geq J(u, x)$.

To this end, we first consider the following second order Taylor expansion: for any $t \geq 0$, there exists a $(\tilde{u}(t), \tilde{x}(t))$ that is a convex combination of $(u(t), x(t))$ and $(\hat{u}(t), \hat{x}(t))$ such that:

$$\begin{aligned} &h(u(t), x(t), t) - h(\hat{u}(t), \hat{x}(t), t) \\ &\quad = \nabla_u h(\hat{u}(t), \hat{x}(t), t)' (u(t) - \hat{u}(t)) + \nabla_x h(\hat{u}(t), \hat{x}(t), t)' (x(t) - \hat{x}(t)) \\ &\qquad + \frac{1}{2} \begin{pmatrix} u(t)' - \hat{u}(t)' & x(t)' - \hat{x}(t)' \end{pmatrix} D^2 h(\tilde{u}(t), \tilde{x}(t), t) \begin{pmatrix} u(t) - \hat{u}(t) \\ x(t) - \hat{x}(t) \end{pmatrix}. \end{aligned}$$

The concavity of h tells us that $D^2 h(\tilde{u}(t), \tilde{x}(t), t)$ is negative semidefinite, so we have

$$\begin{aligned} &h(u(t), x(t), t) - h(\hat{u}(t), \hat{x}(t), t) \\ &\qquad \leq \nabla_u h(\hat{u}(t), \hat{x}(t), t)' (u(t) - \hat{u}(t)) + \nabla_x h(\hat{u}(t), \hat{x}(t), t)' (x(t) - \hat{x}(t)), \end{aligned}$$

and by multiplying both sides by -1,

$$\begin{aligned} &h(\hat{u}(t), \hat{x}(t), t) - h(u(t), x(t), t) \\ &\quad \geq \nabla_u h(\hat{u}(t), \hat{x}(t), t)' (\hat{u}(t) - u(t)) + \nabla_x h(\hat{u}(t), \hat{x}(t), t)' (\hat{x}(t) - u(t)). \end{aligned}$$

We can employ a similar expansion for the components $g_1, \cdots, g_m$ of g, so that

$$\begin{aligned} &g_i(\hat{u}(t), \hat{x}(t), t) - g(u(t), x(t), t) \\ &\qquad \geq \nabla_u g_i(\hat{u}(t), \hat{x}(t), t)' (\hat{u}(t) - u(t)) + \nabla_x g_i(\hat{u}(t), \hat{x}(t), t)' (\hat{x}(t) - x(t)). \end{aligned}$$

for $1 \leq i \leq m$. Putting them all together yields the inequality

$$g(\hat{u}(t), \hat{x}(t), t) - g(u(t), x(t), t)$$

$$\geq D_u g(\hat{u}(t),\hat{x}(t),t)\,(\hat{u}(t)-u(t))+D_x g(\hat{u}(t),\hat{x}(t),t)\,(\hat{x}(t)-x(t))\,,$$

where the inequality is element-wise.

The product rule for differentiation shows us that, for any $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\left(\hat{\lambda}(t)'\left[\hat{x}(t)-x(t)\right]\right) &= \dot{\hat{\lambda}}(t)'\left[\hat{x}(t)-x(t)\right]+\hat{\lambda}(t)'\left[\dot{\hat{x}}(t)-\dot{x}(t)\right] \\
&= \dot{\hat{\lambda}}(t)'\left[\hat{x}(t)-x(t)\right] \\
&\quad +\hat{\lambda}(t)'\left[g(\hat{u}(t),\hat{x}(t),t)-g(u(t),x(t),t)\right]. \\
&\geq \dot{\hat{\lambda}}(t)'\left[\hat{x}(t)-x(t)\right] \\
&\quad +\hat{\lambda}(t)'D_u g(\hat{u}(t),\hat{x}(t),t)\,(\hat{u}(t)-u(t)) \\
&\quad +\hat{\lambda}(t)'D_x g(\hat{u}(t),\hat{x}(t),t)\,(\hat{x}(t)-x(t))\,,
\end{aligned}$$

where the second to last equality follows from the fact that $(\hat{u},\hat{x})$ and (u,x) both satisfy the constraints of the problem, and the last inequality is a reflection of the fact that the entries of $\hat{\lambda}(t)$ are non-negative valued. Integrating both sides from 0 to T yields

$$\begin{aligned}
\hat{\lambda}(T)'\left[\hat{x}(T)-x(T)\right] &\geq \int_0^T \dot{\hat{\lambda}}(t)'\,(\hat{x}(t)-x(t))\,dt \\
&\quad +\int_0^T \hat{\lambda}(t)'D_x g(\hat{u}(t),\hat{x}(t),t)\,(\hat{x}(t)-x(t))\,dt \\
&\quad +\int_0^T \hat{\lambda}(t)'D_u g(\hat{u}(t),\hat{x}(t),t)\,(\hat{u}(t)-u(t))\,dt.
\end{aligned}$$

Using the control equation and the state equation, along with the inequality we obtained from the Taylor expansion of h, we can now see that

$$\begin{aligned}
\hat{\lambda}(T)\left[\hat{x}(T)-x(T)\right] &\geq -\int_0^T \boldsymbol{\nabla}_x h(\hat{u}(t),\hat{x}(t),t)'(\hat{x}(t)-x(t))dt \\
&\quad -\int_0^T \boldsymbol{\nabla}_u h(\hat{u}(t),\hat{x}(t),t)'(\hat{u}(t)-u(t))dt \\
&\geq -\int_0^T h(\hat{u}(t),\hat{x}(t),t)dt+\int_0^T h(u(t),x(t),t)dt,
\end{aligned}$$

or equivalently,

$$\int_0^T h(\hat{u}(t),\hat{x}(t),t)dt - \int_0^T h(u(t),x(t),t)dt \geq \hat{\lambda}(T)'x(T) - \hat{\lambda}(T)'\hat{x}(T)$$
$$\geq -\hat{\lambda}(T)'\hat{x}(T),$$

where the last inequality follows from the observation that $x(T), \hat{\lambda}(T) \in \mathbb{R}^m_+$ for any $T \geq 0$.

Note that, since h is non-negative valued,

$$J(\hat{u},\hat{x}) = \lim_{T\to\infty} \int_0^T h(\hat{u}(t),\hat{x}(t),t)dt$$

by the MCT, and similarly for $J(u,x)$. Therefore, taking $T \to \infty$ on the inequality above and imposing the transversality condition shows us that

$$J(\hat{u},\hat{x}) - J(u,x) \geq 0,$$

as we desired.

Q.E.D.

Note that the proof of the maximum principle is very similar to the theorem in the previous section that showed the Euler equation and transversality condition are sufficient to ensure optimality. This is why we can see Pontryagin's Hamiltonian conditions as the continuous time analogue of the Euler equation in discrete time.

2.4 The Neoclassical Growth Model in Continuous Time

Here we revisit the neoclassical growth model, this time in the context of a continuous time model. We first state the model assumptions, which are almost identical to those of the discrete time model, and then derive the sufficient (and necessary, although we didn't prove that) conditions for maximization.

2.4.1 Model Assumptions

We make the following assumptions:

Commodities There is one consumption good and two inputs, labor and capital. The consumption good and capital are non-negative reals, and labor takes values in the interval $[0, 1]$. This is the same as in the discrete time case.

Preferences There is a single (representative) consumer in the economy who also acts as the representative firm. Their instantaenous utility at time t depends only on the amount of goods they consume $c(t)$; we abstract from labor here and assume that each agent just supplies a fixed unit 1 of their labor. We require c to be a non-negative function, so that

$$\mathcal{U} = C([0, +\infty), \mathbb{R}_+),$$

that is, c is a function on $[0, +\infty)$ taking values in $\mathbb{R}_+$. We can see that this makes $U = \mathbb{R}_+$.

The intertemporal utility given the control c of consumption is defined as

$$W(c) = \int_0^\infty e^{-\rho t} u(c(t)) dt$$

for some discount rate $\rho > 0$ and bounded and twice continuously differentiable instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$. This ensures that

$W(c)$ is non-negative real valued for any choice of $c \in \mathcal{U}$. In addition, as in the model in discrete time, we assume that

$$u'(c) > 0, \quad u''(c) < 0,$$

for any $c > 0$, so that the representative agent experiences positive but diminishing marginal utility.

Technologies The consumption good each period is produced using the inputs of capital and labor hours, where we ignore the latter for now by putting it equal to 1. There exists a production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$y(t) = f(k(t))$$

for any $t \in \mathbb{N}$, where $y(t)$ is the amount of consumption goods produced at time t and $k(t)$ is the amount of capital supplied at time t. The household then either consumes $y(t)$ or invests it:

$$c(t) + i(t) = y(t).$$

Finally, investment determines the dynamics of capital, where $\delta \in (0,1)$ is the depreciation rate of capital:

$$\dot{k}(t) = i(t) - \delta \cdot k(t).$$

Putting these conditions together, we can see that the law of motion of capital is given as

$$\dot{k}(t) = f(k(t)) - \delta \cdot k(t) - c(t)$$

for any $t \geq 0$. Capital thus becomes the state variable in our model. As in the discrete time model, we require there to be a non-negative amount of capital each period, so that $\mathcal{X} = C([0, +\infty), \mathbb{R}_+)$. This makes $X = \mathbb{R}_+$.

We assume that the production function f is twice continuously differentiable, with the properties

$$f'(k) > 0, \quad f''(k) < 0$$

for any $k \in \mathbb{R}_+$, so that the marginal product for capital is positive and diminishing.

Finally, we assume that the the following Inada conditions hold:

$$\lim_{k \downarrow 0} f'(k) = +\infty$$
$$\lim_{k \to \infty} f'(k) = 0.$$

The representative household's problem is equivalent to the social planner's problem under the feasibility constraint:

$$\max_{c \in \mathcal{U}} \quad \int_0^\infty e^{-\rho t} u(c(t)) dt$$

$$\text{subject to} \quad k(0) = k_0, \quad \dot{k}(t) = f(k(t)) - \delta \cdot k(t) - c(t) \quad \text{for any } t \geq 0$$

given the initial capital stock $k_0 > 0$. To apply the maximum principle, we first verify that the functions involved in the above problem satisfy the conditions of the theorem.

2.4.2 Solving for the Optimal Paths

In the neoclassical growth model, the felicity function and constraint functions are given as

$$h(c,k,t) = e^{-\rho t}u(c)$$
$$g(c,k,t) = f(k) - \delta k - c$$

for any $t \geq 0$, $c \in U$ and $k \in X$. For the maximum principle to work, h and g must be twice continuously differentiable concave functions. By the twice continuous differentiability of u and F, h and g are clearly twice continuously differentiable. Furthermore, the concavity of u tells us that h is also concave.

Likewise, the concavity of f shows us that g is concave in (c,k). For any $(c,k) \in U \times X$,

$$D^2 g(c,k,t) = \begin{pmatrix} 0 & 0 \\ 0 & f''(k) \end{pmatrix}.$$

The concavity of f shows us that $f''(k)$ is negative, which implies that $D^2 g(c,k,t)$ is negative semidefinite. Therefore, g is concave with respect to (c,k).

We can now apply the maximum principle to find a solution to the dynamic optimization problem above. The Hamiltonian of the problem is given as

$$H(c,k,\lambda,t) = e^{-\rho t}u(c) + \lambda \cdot [f(k) - \delta k - c]$$

for any $(c,k,\lambda,t) \in U \times X \times \mathbb{R}_+ \times [0,+\infty)$. Suppose there exist continuously differentiable functions $c \in \mathcal{U}$, $k \in \mathcal{X}$, and $\lambda \in C([0,+\infty),\mathbb{R}_+)$ such that the conditions of the maximum principle are satisfied:

$$e^{-\rho t}u'(c(t)) - \lambda(t) = 0 \qquad \text{(Control Equation)}$$

$$\lambda(t)\left[f'(k(t)) - \delta\right] = -\dot{\lambda}(t) \qquad \text{(State Equation)}$$

$$f(k(t)) - \delta k(t) - c(t) = \dot{k}(t) \qquad \text{(Costate Equation)}$$

$$\lim_{T\to\infty} \lambda(T)k(T) = 0. \qquad \text{(Transversality Condition)}$$

Solving the control equations shows us that

$$\lambda(t) = e^{-\rho t} u'(c(t))$$

so that

$$\dot{\lambda}(t) = -\rho \cdot e^{-\rho t} u'(c(t)) + e^{-\rho t} u''(c(t)) \dot{c}(t).$$

Substituting these into the state equation yields

$$\rho \cdot e^{-\rho t} u'(c(t)) - e^{-\rho t} u''(c(t)) \dot{c}(t) = e^{-\rho t} u'(c(t)) \left[f'(k(t)) - \delta \right],$$

or equivalently,

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{u''(c(t))c(t)} \left[f'(k(t)) - \delta - \rho \right].$$

This is the continuous time analogue of the Eulere equation; note that $\frac{1}{\sigma(c(t))} = -\frac{u'(c(t))}{u''(c(t))c(t)}$ is the reciprocal of the relative risk aversion coefficient at time t. Together with the law of motion for capital and the transversality condition, the Euler equation characterizes the dynamics of the economy.

The steady state values of capital and consumption (k^*, c^*) are the values at which capital nor consumption evolve further. Specifically, they satisfy

$$\begin{aligned} c^* &= f(k^*) - \delta k^* \\ f'(k^*) &= \delta + \rho. \end{aligned}$$

The three equations

$$\dot{k}_t = f(k(t)) - \delta k(t) - c(t)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma(c(t))} \left[f'(k(t)) - \delta - \rho \right]$$

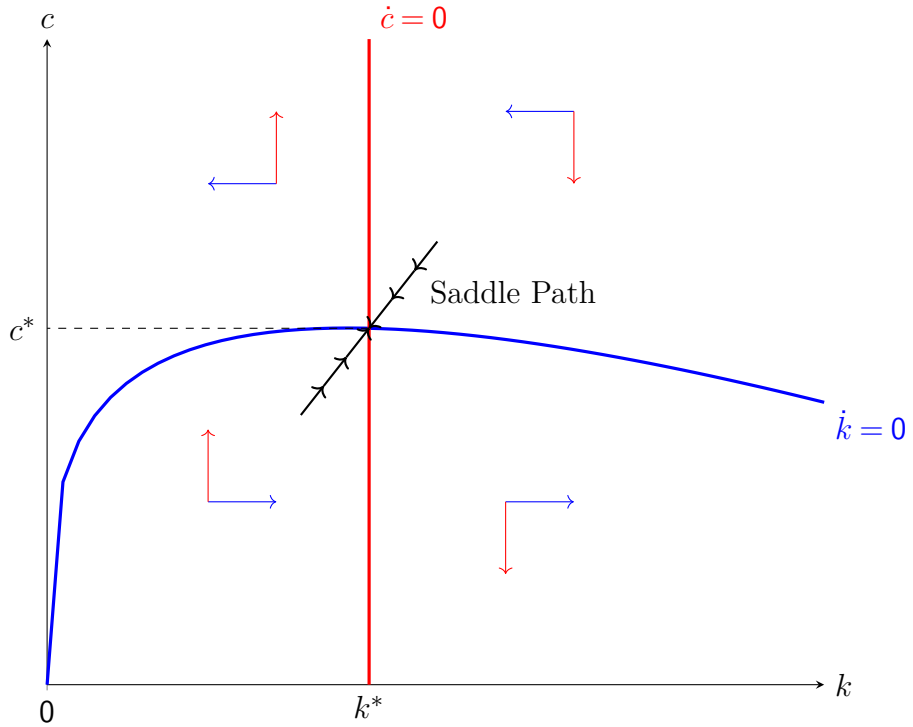
$$\lim_{T \to \infty} e^{-\rho T} u'(c(T)) k(T) = 0$$

allows us to obtain the unique saddle path that heads to the steady state (k^*, c^*).

If $c(t) < f(k(t)) - \delta k(t)$, then $0 < k'(t) = f(k(t)) - \delta k(t) - c(t)$; this tells us that $k(t)$ is increasing underneath the locus $k'(t) = 0$ and, conversely, decreasing above it. Likewise, if $k(t) < k^*$, then

$$f'(k(t)) - \delta - \rho > f'(k^*) - \delta - \rho = 0$$

because f' is decreasing, and therefore $c(t) > 0$. In other words, $c(t)$ is increasing to the left of the locus $c'(t) = 0$ and decreasing on its right. These movements are summarized in the graph below, and together with the transversality, they suggest that there exists a unique initial value $c(0)$ for consumption such that the economy travels along the saddle path to the steady state (k^*, c^*):



Specifically, letting c_0^* be the unique initial value of consumption that places the economy on the saddle path, then if $c_0 > c_0^*$, consumption increases and capital decreases until capital drops to 0. At that point, the consumer must choose an infinite amount of consumption from no income, which violates the resource (feasibility) constraint. On the other hand, if $c_0 < c_0^*$, then consumption drops to 0 while capital continually increases. Since capital cannot rise above the maximum maintainable capital stock $\hat{k}$, while the marginal utility of consumption diverges to $+\infty$ due to the Inada conditions, this leads

to a violation of the transversality condition. This shows that the only optimal path in the model is the saddle path.

2.4.3 Transition Dynamics

Given the laws of motion for capital and consumption in this model, we want to investigate how the economy transitions to the steady state. There are two main ways that we can approach this problem; through linearization around the steady state or by examining the slope of the saddle path. For the sake of simplicity, consider the case where the instantaneous utility function is given in the CRRA form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

for any $c \geq 0$, where $\sigma < 1$ is the (constant) relative risk aversion coefficient. Under the intertemporally separable utiltiy function specification in the neoclassical growth model, this parameter is closely related to the elasticity of intertemporal substitution, which is defined as

$$IES_{t,s} = \left| \frac{\partial \log\left(\frac{c_t}{c_s}\right)}{\partial \log\left(\frac{u'(c_t)}{u'(c_s)}\right)} \right|.$$

In other words, $IES_{t,s}$ measures the degree to which the representative household is willing to decrease consumption at time t and increase consumption at time s, that is, substitute the consumption of time t with consumption at time s, when the relative price of consumption at time t to consumption at time s, measured in terms of their relative marginal utilites, increases. Under CRRA utility, this value is given as

$$IES = \left| \frac{\partial \log\left(\frac{c_t}{c_s}\right)}{\partial \log\left(\left(\frac{c_t}{c_s}\right)^{-\sigma}\right)} \right| = \frac{1}{\sigma}.$$

In other words, the more risk averse the representative household is, that is, the larger σ, the lower the intertemporal elasticitiy of substitution. This indicates how, if a consumer is more risk averse, they tend to prefer risk-free paths of consumption over paths of consumption higher volatility. That is to say, they choose to smooth their consumption

over time to a higher degree, so that changes in relative marginal utility across different periods are less likely to affect relative consumption between those periods.

Linearization around Steady State

When the economy is close to the steady state, linearizing around the steady state proves to be an adequate approximation of the transition dynamics of the model. The dynamics of the growth model are summarized by the following two-dimensional system of non-linear ordinary differential equations:

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma}\left[f'(k(t)) - \delta - \rho\right] c(t) \\ \\ f(k(t)) - \delta k(t) - c(t) \end{pmatrix}.$$

Defining $g : \mathbb{R}^2_{++} \to \mathbb{R}^2$ as

$$g(c,k) = \begin{pmatrix} \frac{1}{\sigma}\left[f'(k) - \delta - \rho\right] c \\ \\ f(k) - \delta k - c \end{pmatrix}$$

for any $(c,k) \in \mathbb{R}^2_{++}$, and letting $z(t) = (c(t), k(t))'$, we can see that the above system can be written succinctly as

$$\dot{z}(t) = g(z(t)).$$

A first order Taylor approximation of $z(t)$ around the steady state $z^* = (c^*, k^*)'$ yields

$$\dot{z}(t) = g(z(t)) \approx g(z^*) + Dg(z^*)(z(t) - z^*),$$

where

$$Dg(z^*) = \begin{pmatrix} \frac{\partial g_1(z^*)}{\partial c} & \frac{\partial g_1(z^*)}{\partial k} \\ \\ \frac{\partial g_2(z^*)}{\partial c} & \frac{\partial g_2(z^*)}{\partial k} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma}\left[f'(k^*) - \delta - \rho\right] c^* & \frac{f''(k^*)}{\sigma} \\ \\ -1 & f'(k*) - \delta \end{pmatrix}.$$

Using the fact that $f'(k^*) - \delta - \rho = 0$, we can express the Jacobian above as

$$Dg(z^*) = \begin{pmatrix} 0 & \frac{f''(k^*)}{\sigma} c^* \\ -1 & \rho \end{pmatrix}.$$

It follows that the linearized system is

$$\begin{pmatrix} \dot{c}(t) - c^* \\ \dot{k}(t) - k^* \end{pmatrix} = \begin{pmatrix} 0 & \frac{f''(k^*)}{\sigma} c^* \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) - c^* \\ k(t) - k^* \end{pmatrix}.$$

This is a two-dimensional system of linear differential equations, so we can see that the convergence of the system depends on the eigenvalues of the matrix $Dg(z^*)$; this is formalized in the following theorem.

Lemma (Convergence Properties of Linear ODEs)
Consider an n-dimensional system of linear differential equations

$$y'(t) = Ay(t),$$

where $y : \mathbb{R}_+ \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a matrix with n distinct eigenvalues $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. If $k \leq n$ of the eigenvalues have negative real parts, then there exists a k-dimensional subspace L of $\mathbb{R}^n$ such that, for any initial value $y(0) \in L$, there exists a unique solution $\hat{y}(t)$ to the differential equation above that converges to $\mathbf{0}$ as $t \to \infty$.

Proof) Since A has n distinct eigenvalues, it is diagonalizable. That is, there exists an invertible matrix $P \in \mathbb{R}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that the diagonal elements of D are equal to the eigenvalues of A, the columns $v_1, \cdots, v_n \in \mathbb{R}^n$ of P are eigenvectors corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_n$, and

$$A = PDP^{-1}.$$

Then, the system can be rewritten as

$$P^{-1}y'(t) = D \cdot \left[P^{-1}y(t)\right].$$

Defining $z(t) = P^{-1}y(t)$, we can see that this reduces to

$$z'(t) = Dz(t) = \begin{pmatrix} \lambda_1 z_1(t) \\ \vdots \\ \lambda_n z_n(t) \end{pmatrix}.$$

For each $1 \leq i \leq n$, we thus have $z_i'(t) = \lambda_i z_i(t)$ and therefore

$$z_i(t) = \exp(\lambda_i t) z_i(0).$$

Substituting this back into the original system, we can see that

$$z(t) = \begin{pmatrix} \exp(\lambda_1 t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \exp(\lambda_n t) \end{pmatrix} z(0) = \exp(Dt) z(0),$$

so that

$$\begin{aligned} y(t) &= P \exp(Dt) P^{-1} \cdot y(0) \\ &= \begin{pmatrix} \exp(\lambda_1 t) v_1 & \cdots & \exp(\lambda_n t) v_n \end{pmatrix} P^{-1} \cdot z(0) \\ &= \sum_{j=1}^{n} c_j v_j \cdot \exp(\lambda_j t) \end{aligned}$$

for the vector $c = (c_1, \cdots, c_n)' = P^{-1}y(0)$.

For any $z \in \mathbb{C}$, we can see that

$$\begin{aligned} \exp(zt) &= \exp(Re(z)t) \cdot \exp(i \cdot Im(z)t) \\ &= \exp(Re(z)t) \cdot [\cos(Im(z)t) + i \cdot \sin(Im(z)t)] \end{aligned}$$

by Euler's formula. Since the cosine and sine functions are bounded, this implies that

$$\lim_{t \to \infty} \exp(zt) = 0 \quad \text{if and only if} \quad Re(z) < 0.$$

Suppose the eigenvalues are arranged so that the first k eigenvalues have negative

real parts, while the last $n-k$ do not. In this case, for $y(t)$ to converge to $\mathbf{0}$ as $t \to \infty$, we must have $c_{k+1} = \cdots = c_n = 0$, so that

$$y(t) = \sum_{j=1}^{k} c_j v_j \cdot \exp(\lambda_j t).$$

This reduces our search for initial values to the search of the set of $x \in \mathbb{R}^n$ for which the last $n-k$ rows of $P^{-1}x$ equal 0; but this is simply the set of all points of the linear subspace

$$L = \operatorname{span}(\{v_1, \cdots, v_k\}),$$

where L is clearly a k-dimensional space due to the linear independence of $v_1, \cdots, v_k$. Therefore, for any initial value $y(0) \in L$, since the last $n-k$ elements of $P^{-1}y(0)$ are equal to 0, there exists a unique solution to the system $y' = Ay$ in the form of

$$y(t) = \sum_{j=1}^{k} c_j v_j \cdot \exp(\lambda_j t),$$

where $c = (c_1, \cdots, c_k)'$ collects the first k elements of $P^{-1}y(0)$.

Q.E.D.

Returning to the neoclassical growth model, note that the eigenvalues of the matrix $Dg(z^*)$ are obtained as the solutions to the characteristic equation

$$\det(Dg(z^*) - \lambda \cdot I_2) = \det \begin{pmatrix} -\lambda & \frac{f''(k^*)}{\sigma} c^* \\ -1 & \rho - \lambda \end{pmatrix} = \lambda^2 - \rho\lambda + \frac{f''(k^*)}{\sigma} c^* = 0.$$

We can now see that the eigenvalues of $Dg(z^*)$ are given as

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - \frac{4f''(k^*)c^*}{\sigma}}}{2} < 0 < \frac{\rho + \sqrt{\rho^2 - \frac{4f''(k^*)c^*}{\sigma}}}{2} = \lambda_2,$$

where λ_1, λ_2 are real and distinct because $f''(k^*) < 0$ and thus

$$\rho^2 - \frac{4f''(k^*)c^*}{\sigma} > 0.$$

Let $v_1, v_2 \in \mathbb{R}^2$ be corresponding eigenvectors, and define $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$. Then, by the lemma above, we can see that the initial values (c_0, k_0) that situate the economy on the saddle path must be chosen so that (c_0, k_0) is be a scalar multiple of v_1. Since k_0 is given, this allows us to uniquely pin down c_0. Specifically,

$$c_0 = a \cdot v_{11}, \quad k_0 = a \cdot v_{12}$$

for some $a \in \mathbb{R}$, so $a = \frac{k_0}{v_{12}}$ and thus $c_0 = \frac{v_{11}}{v_{12}} k_0$. The linearized paths to the steady state are now given as

$$\begin{aligned} c(t) &\approx c^* + \left(\frac{v_{11}}{v_{12}} k_0 - c^* \right) \cdot \exp(\lambda_1 t) \\ k(t) &\approx k^* + (k_0 - k^*) \cdot \exp(\lambda_1 t). \end{aligned}$$

This formulation allows us, among other things, to find the half life of $k(t)$, that is, the amount of time it takes for $k(t)$ to travel halfway from k_0 to k^*. Letting the half-life be denoted $t_{1/2}$, note that, at $t_{1/2}$,

$$\frac{1}{2}(k_0 - k^*) = k(t_{1/2}) - k^* = (k_0 - k^*) \exp\left(\lambda_1 \cdot t_{1/2} \right).$$

Therefore,

$$t_{1/2} = -\log(2) \cdot \frac{1}{\lambda_1} = \frac{\log(2)}{|\lambda_1|}.$$

The larger $|\lambda_1|$, the faster $k(t)$ grows to k^* and therefore the shorter the half-life.

Slope of the Saddle Path at Steady State

Another way to appraoch the transition dynamics of the model is to study the slope of the saddle path. The saddle path can be viewed as the path of some policy function, that is, as the function $C : \mathbb{R}_+ \to \mathbb{R}$ such that $C(k)$ is the optimal amount of consumption the representative household chooses given the amount of capital k. In this context, the optimal initial consumption c_0^* would be given as $c_0^* = C(k_0)$, and for any $t \geq 0$ we would have $c(t) = C(k(t))$.

The chain rule then implies that $C'(k(t))k'(t) = c'(t)$, or

$$\begin{aligned} C'(k(t)) = \frac{\dot{c}(t)}{\dot{k}(t)} &= \frac{\frac{1}{\sigma}\left[f'(k(t)) - \delta - \rho\right] c(t)}{f(k(t)) - \delta k(t) - c(t)} \\ &= \frac{\frac{1}{\sigma}\left[f'(k(t)) - \delta - \rho\right] C(k(t))}{f(k(t)) - \delta k(t) - C(k(t))} \end{aligned}$$

for any $t \geq 0$. By implication, the slope of the saddle path at any point $k \geq 0$ is given as

$$C'(k) = \frac{\frac{1}{\sigma}\left[f'(k) - \delta - \rho\right] C(k)}{f(k) - \delta k - C(k)}.$$

This is a non-linear ODE, and can be solved through numerical methods (no analytic method exists yet).

Instead of solving the above ODE, we can just study the slope of the saddle path at the steady state level of capital k^*, to see how it changes in response to a shift in the steady state. This is useful because it allows us to study how changes in model parameters affect the transition dynamics of the model.

From the formula above, we can see that we cannot evaluate $C'(\cdot)$ directly at k^*, since both the numerator and denominator become 0. As such, we obtain $C'(k^*)$ as the limit of $C'(k)$:

$$\begin{aligned} C'(k^*) = \lim_{k \to k^*} C'(k) &= \lim_{k \to k^*} \frac{\frac{1}{\sigma}\left[f'(k) - \delta - \rho\right] C(k)}{f(k) - \delta k - C(k)} \\ &= \lim_{k \to k^*} \frac{\frac{1}{\sigma} f''(k) C(k) + \frac{1}{\sigma}\left[f'(k) - \delta - \rho\right] C'(k)}{f'(k) - \delta - C'(k)} \\ &= \frac{\frac{1}{\sigma} f''(k^*) c^*}{f'(k^*) - \delta - C'(k^*)} = \frac{\frac{1}{\sigma} f''(k^*) c^*}{\rho - C'(k^*)}, \end{aligned}$$

where we applied L'Hospital's rule. $C'(k^*)$ can now be obtained as the (positive) solution to the quadratic equation

$$\lambda^2 - \rho\lambda + \frac{f''(k^*)c^*}{\sigma} = 0.$$

Surprisingly, this is exactly the same equation as the characteristic equation for the linearized ODE we studied above. There, we focused on the negative eigenvalue λ_1 and its eigenvector as the direction in which the system converges, but here we shift our focus to the positive eigenvalue λ_2 as the slope of the saddle path at the steady state:

$$C'(k^*) = \lambda_2 = \frac{\rho + \sqrt{\rho^2 - \frac{4f''(k^*)c^*}{\sigma}}}{2}.$$

A Transition Experiment: Changes in TFP

As an example of how the quantity $C'(k^*)$ can be used to study the transition dynamics of the economy, consider the functional form

$$f(k) = \varepsilon k^{\alpha}$$

for the production function, where $\alpha \in (0,1)$ and $\varepsilon > 0$ represents total factor productivity. Then, the laws of motion of the economy become

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[\alpha\varepsilon k(t)^{\alpha-1} - \delta - \rho\right]$$

$$\dot{k}(t) = \varepsilon k(t)^{\alpha} - \delta k(t) - c(t)$$

$$\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0.$$

The steady state values of capital and consumption can now be written as functions of ε:

$$k^*(\varepsilon) = \left(\frac{\alpha\varepsilon}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$$

$$c^*(\varepsilon) = \varepsilon(k^*)^{\alpha} - \delta k^* = \varepsilon^{\frac{1}{1-\alpha}} \cdot \frac{\delta(1-\alpha)+\rho}{\delta+\rho}\left(\frac{\alpha}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}}.$$

$k^*(\cdot)$ and $c^*(\cdot)$ are both increasing functions of ε, but the ratio $m = \frac{c^*(\varepsilon)}{k^*(\varepsilon)}$ is not dependent on ε. This shows us that, for any $\varepsilon > 0$, the pair $(k^*(\varepsilon), c^*(\varepsilon))$ lies on the line $c = mk$.

Similarly, the slope of the saddle path at the steady state does not depend on ε. To see this, note that

$$\begin{aligned}
\frac{f''(k^*)}{\sigma}c^* &= \frac{\alpha(\alpha-1)\varepsilon(k^*)^{\alpha-2}}{\sigma}c^* \\
&= \varepsilon^{1+\frac{\alpha-2}{1-\alpha}+\frac{1}{1-\alpha}} \cdot \frac{\alpha(\alpha-1)}{\sigma}\left(\frac{\alpha}{\delta+\rho}\right)^{\frac{\alpha-2}{1-\alpha}} \frac{\delta(1-\alpha)+\rho}{\delta+\rho}\left(\frac{\alpha}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}} \\
&= -\frac{(\delta+\rho)(1-\alpha)}{\sigma\alpha}\left(\delta(1-\alpha)+\rho\right).
\end{aligned}$$

It follows that

$$C'(k^*) = \frac{\rho + \sqrt{\rho^2 - \frac{4f''(k^*)c^*}{\sigma}}}{2}$$

is also independent of ε. This allows us to consider two cases:

Case 1 $C'(k^*) > m$

In this case, in response to any increase in ε, the economy must converge to the new steady state from underneath the line $c = mk$. In other words, consumption must instantaneously decrease as soon as ε increases, in order to move to the new saddle path.

This decrease in consumption at the time of the increase in TFP reflects the fact that, in this case, the substitution effect dominates the income effect. Specifically, an increase in ε at time t_0 pulls the economy in two directions. First, production at time t_0 increases, so that the representative household can consume more at time t_0. On the other hand, an increase in ε raises the marginal product of capital, so that saving at time t_0 could lead to greater returns to saving in the future; heuristically, this has the effect of making future consumption seem cheaper, and drives down current consumption. The latter is the substitution effect, and the former the income effect. If $C'(k^*) > m$, then consumption decreases at time t_0 as agents choose to save more now to consume more later; thus, in this case the substitution effect dominates the income effect.

Case 2 $C'(k^*) < m$

In this case, in response to any increase in ε, the economy converges to the new steady state from above the line $c = mk$, so that consumption at the time t_0 of the change in TFP must increase. Thus, in this case the income effect dominates the substitution effect.

To further investigate which parameters determine whether the income or substitution effect dominates, that is, whether $C'(k^*)$ is greater or lower than m, we return to the expression for the slope of the saddle path at the steady state:

$$C'(k^*) = \frac{\rho + \sqrt{\rho^2 - \frac{4f''(k^*)c^*}{\sigma}}}{2}.$$

We want to find the conditions under which $C'(k^*) > m = \frac{c^*}{k^*}$. Note that

$$\frac{c^*}{k^*} = \frac{\delta(1-\alpha)+\rho}{\alpha},$$

and we already saw that

$$\frac{f''(k^*)}{\sigma}c^* = -\frac{(\delta+\rho)(1-\alpha)}{\sigma\alpha}\left(\delta(1-\alpha)+\rho\right),$$

so we want to find necessary and sufficient conditions for

$$C'(k^*) = \frac{\rho+\sqrt{\rho^2+\frac{4(\delta+\rho)(1-\alpha)}{\sigma\alpha}\left(\delta(1-\alpha)+\rho\right)}}{2} > \frac{\delta(1-\alpha)+\rho}{\alpha} = \frac{c^*}{k^*}.$$

To this end, first multiply both sides by 2 and subtract ρ from both sides:

$$\sqrt{\rho^2+\frac{4(\delta+\rho)(1-\alpha)}{\sigma\alpha}\left(\delta(1-\alpha)+\rho\right)} > \frac{(2-\alpha)\rho+2(1-\alpha)\delta}{\alpha}.$$

Squaring both sides and multiplying both sides by $\sigma\alpha^2$ shows us that

$$\begin{aligned}\sigma\alpha^2\rho^2+4\alpha(\delta+\rho)(1-\alpha)^2\delta+4\alpha(\delta+\rho)(1-\alpha)\rho&\\ > \sigma(2-\alpha)^2\rho^2+4\sigma(2-\alpha)(1-\alpha)\delta\rho+4\sigma(1-\alpha)^2\delta^2.&\end{aligned}$$

Rearranging and regrouping terms on the left hand side reveals that this inequality is equivalent to

$$\begin{aligned}\alpha(\alpha\sigma-4\alpha+4)\rho^2+4\alpha(2-\alpha)(1-\alpha)\delta\rho+4\alpha(1-\alpha)^2\delta^2&\\ > \sigma(2-\alpha)^2\rho^2+4\sigma(2-\alpha)(1-\alpha)\delta\rho+4\sigma(1-\alpha)^2\delta^2.&\end{aligned}$$

Finally, moving all terms on the right hand side to the left hand side and dividing by $4(1-\alpha)$ yields

$$(\alpha-\sigma)\rho^2+(\alpha-\sigma)(2-\alpha)\delta\rho+(\alpha-\rho)(1-\alpha)\delta^2 > 0.$$

Clearly, this inequality holds if and only if $\alpha > \sigma$. In other words, the substitution effect dominates the income effect of an increase in TFP if and only if the capital share α is larger than the relative risk aversion coefficient, or inverse intertermporal elasticity of

substitution σ.

Heuristically, a large σ indicates a greater desire to smooth consumption. Thus, when $\sigma > \alpha$ the representative household quickly raises consumption to near the level of consumption at the new steady state, so that consumption is less volatile as the economy converges to the new steady state.

On the other hand, a large α indicates a greater marginal product of capital, or greater returns to saving. This is why, if $\alpha > \sigma$, then the representative household chooses to forgo current consumption in favor of saving more today and reaping greater returns in the future.

Chapter 3

Markov Kernels and Chains

So far, we have discussed deterministic macroeconomic models, that is, models without any uncertainty. When introducing uncertainty into macroeconomic models, we want to do so in a manner that preserves the recursive structure of the deterministic problems. To this end, we focus on stochasticity inducd by Markov processes, which are stochastic processes whose current value depends only on yesterday's value. This allows us to reduce an infinite horizon model into a two-period recursive problem, as we will see below.

This chapter is dedicated to the study of the core properties of Markov kernels and chains. Among other things, we define what is meant by a Markov chain, and furnish conditions under which Markov chains converge regardless of the initial distribution. These limiting, or stationary, distributions will serve as the stochastic analogue of fthe deterministic notion of the steady state.

3.1 Core Concepts

We first introduce the core concepts pertaining to Markov chains. Markov chains are formalized through the use of Markov kernels, which are a particular type of transition probability kernel, or stochastic map. The properties of Markov chains can then be summarized through this kernel, and we discuss some important properties in the next section.

3.2 Transition Probabiltiy Kernels

A transition probability kernel K from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ is a function $K : E \times \mathcal{F} \to [0, 1]$ such that the mapping

$$x \mapsto K(x, A)$$

is a $\mathcal{E}$-measurable non-negative function for any $A \in \mathcal{F}$, and the mapping

$$A \mapsto K(x, A)$$

is a probability measure on $(F, \mathcal{F})$ for any $x \in E$.

Since the sections of a transition probability kernel can be viewed as either a function or a probability measure, we can think of both integrating a transition probability kernel and integrating with respect to the same kernel. We study each in turn.

Kernel-Function Operators

Given a transition probability kernel K from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ and a non-negative or bounded function $f : E \times F \to \mathbb{R}$ that is $\mathcal{E} \bigotimes \mathcal{F}$-measurable, for any $x \in E$ the integral of $f(x, \cdot)$ with respect to the probability measure $K(x, \cdot)$ for some $x \in E$ is denoted

$$\int_Z f(x, y) K(x, dy).$$

whenever the integral is well-defined. Defining the operation T_K as

$$[T_K(f)](x) = \int_Z f(x, y) K(x, dy).$$

for any $f \in (\mathcal{E} \bigotimes \mathcal{F})_{+,b}$ and $x \in E$, we can show that $T_K(f)$ is a non-negative/bounded $\mathcal{E}$-measurable function. The operator $T_K : (\mathcal{E} \bigotimes \mathcal{F})_{+,b} \to \mathcal{E}_{+,b}$ is called a kernel-function operator, and it will appear very often in the theory of discrete Markov processes.

Measure-Kernels

Now let μ be a measure on $(E, \mathcal{E})$. Then, the function $\pi : \mathcal{E} \bigotimes \mathcal{F} \to [0, 1]$ defined as

$$\pi(A) = \int_Z T_K(I_A) d\mu$$

for any $A \in \mathcal{E} \bigotimes \mathcal{F}$ is a measure on $(E, \mathcal{E}) \bigotimes (F, \mathcal{F})$, and it is the unique meaure on this product space such that

$$\pi(A \times B) = \int_A K(x, B) d\mu(x)$$

for any $A \in \mathcal{E}$ and $B \in \mathcal{F}$. Furthermore,

$$\int_{E \times F} f d\pi = \int_Z [T_K(f)](x) d\mu(x)$$

for any $f \in (\mathcal{E} \bigotimes \mathcal{F})_+$. We denote π by the product $\mu \times K$, or sometimes simply by μK, and call it a measure-kernel.

3.2.1 Markov Kernels

A Markov kernel with state space $(Z, \mathcal{Z})$ is a transition probability kernel P from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})$. We define the transformation $T_P : \mathcal{Z}_{+,b} \to \mathcal{Z}_{+,b}$ as

$$[T_P(f)](z) = \int_Z f(z') P(z, dz')$$

for any $z \in Z$ and $f \in \mathcal{Z}_{+,b}$, where $\mathcal{Z}_{+,b}$ is the set of all non-negative or bounded $\mathcal{Z}$-measurable functions. The above discussion shows us that $T_P(f)$ is measurable, and non-negativity/boundedness are clearly preserved under the transformation T_P.

Below we introduce some operations we can perform with Markov kernels.

Identity Kernels

We define the identity kernel I from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})$ as

$$I(z, A) = \delta_z(A) \quad \text{for any } (z, A) \in Z \times \mathcal{Z},$$

where δ_z is the Dirac measure on $(Z, \mathcal{Z})$ sitting at $z \in Z$. For any $A \in \mathcal{Z}$, $I(\cdot, A) = I_A \in \mathcal{Z}_+$, while for any $z \in A$, $I(z, \cdot) = \delta_z$ is a probability measure on $(Z, \mathcal{Z})$, so that I is a well-defined Markov kernel on $(Z, \mathcal{Z})$. It is called the identity kernel because

$$[T_I(f)](z) = \int_Z f(z')I(z, dz') = \int_Z f(z')d\delta_z(z') = f(z)$$

for any $\mathcal{Z}$-measurable real valued function f and $z \in Z$, so that $T_I(f) = f$.

Product of Markov Kernels

Given two Markov kernels P, Q on $(Z, \mathcal{Z})$, we define the product PQ of P and Q as

$$(PQ)(z, A) = \int_Z Q(z', A)P(z, dz')$$

for any $(z, A) \in Z \times \mathcal{Z}$. By the remark 1 above, $(PQ)(\cdot, A)$ is a bounded non-negative $\mathcal{Z}$-measurable function, and by remark 2, $(PQ)(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$, so that PQ is a Markov kernel on $(Z, \mathcal{Z})$. Note that, if $Q = I$, then

$$(PI)(z, A) = \int_Z I(z', A)P(z, dz') = \int_Z I_A(z')P(z, dz') = P(z, A)$$

and

$$(IP)(z, A) = \int_Z P(z', A)I(z, dz') = \int_Z P(z', A)d\delta_z(z') = P(z, A),$$

so that $PI = IP = P$.

Now we define $P^{\mathsf{0}} = I$ and, for any $t \in N_+$, P^t is defined inductively as $P^t = P^{t-1}P$. By definition, $P^{\mathsf{1}} = P^{\mathsf{0}}P = P$, and

$$P^t(z, A) = \int_Z P(z', A)P^{t-1}(z, dz')$$

for any $(z, A) \in Z \times \mathcal{Z}$. Choose any real-valued measurable function $f : Z \to \mathbb{R}$, and fix $z \in Z$. Denoting the probability measure $P^{t-1}(z, \cdot) = \mu_z$, we can view $P^t(z, \cdot)$ as the measure-kernel $\mu_z P$. As such,

$$\int_Z f(z')P^t(z, dz') = \int_Z f d(\mu_z P) = \int_Z T_P(f) d\mu_z := \int_Z [T_P(f)](z')P(z, dz').$$

3.3 Properties of Markov Kernels

Here we introduce some properties of Markov kernels that will prove useful later on. The two main properties we consider here are the Feller property and monotonicity. These can be understood to be properties that allow the mapping T_P that transforms a $\mathcal{Z}$-measurable funciton into another $\mathcal{Z}$-measurable function to preserve certain properties; the Feller property allows T_P to preserve continuity, while monotonicity allows it to preserve (obviously) monotonicity.

3.3.1 The Feller Property

The Feller property holds for a general class of Markov kernels. Suppose Z is metrizable under the metric ρ, and let $\mathcal{Z}$ be the Borel σ-algebra induced by ρ. Assume further that (Z, ρ) is a complete and separable metric space. Let $\mathcal{P}$ be the Prohorov metric on $\mathcal{M}_p(\mathcal{Z})$, the space of all probability measures on $(Z, \mathcal{Z})$. Since (Z, ρ) is separable, $\mathcal{P}$ induces the topology of weak convergence on $\mathcal{M}_p(\mathcal{Z})$ and the metric space $(\mathcal{M}_p(\mathcal{Z}), \mathcal{P})$ is itself separable.

We say that the Markov kernel P on $(Z, \mathcal{Z})$ has the Feller property if the mapping

$$z \mapsto P(z, \cdot)$$

from Z into $\mathcal{M}_p(\mathcal{Z})$ is continuous with respect to the metrics ρ and $\mathcal{P}$. In light of the fact that $\mathcal{P}$ induces the topology of weak convergence on $\mathcal{M}_p(\mathcal{Z})$, this implies that, for any bounded and real continuous function $f \in C_b(Z, \mathbb{R})$, and a sequence $\{z_n\}_{n \in N_+}$ in Z converging in ρ to $z \in Z$, we have

$$\lim_{n \to \infty} \int_Z f(z') P(z_n, dz') = \int_Z f(z') P(z, dz').$$

Another way to write it is that

$$\lim_{n \to \infty} [T_P(f)](z_n) = [T_P(f)](z)$$

for any $f \in C_b(Z, \mathbb{R})$ and sequence $\{z_n\}_{n \in N_+}$ that converges to $z \in Z$. Therefore, under the Feller property, T_P maps $C_b(Z, \mathbb{R})$ into itself.

The next result shows that the product of any Markov kernel with the Feller property also has the Feller property:

Lemma (Preservation of Feller Property across Kernel Products)
Let P be a Markov kernel on $(Z, \mathcal{Z})$, where (Z, ρ) is a complete and separable metric space. If P has the Feller property, then for any $t \in N_+$, the product P^t is a Markov kernel on $(Z, \mathcal{Z})$ that also has the Feller property.

Proof) Suppose P has the Feller property. Then, the claim above holds for $t = 1$, so suppose now that P^t has the Feller property for some $t \geq 1$. Then, for any continuous and bounded function $f \in C_b(Z, \mathbb{R})$, and a sequence $\{z_n\}_{n \in N_+}$ in Z that converges to some $z \in Z$,

$$\int_Z f(z') P^{t+1}(z_n, dz') = \int_Z [T_P(f)](z') P^t(z_n, dz')$$

for any $n \in N_+$. Since $T_P(f)$ is a continuous and bounded function on Z because T_P preserves continuity under the Feller property, and it follows from the inductive hypothesis that P^t has the Feller property, we can conclude that

$$\begin{aligned} \lim_{n \to \infty} \int_Z f(z') P^{t+1}(z_n, dz') &= \lim_{n \to \infty} \int_Z [T_P(f)](z') P^t(z_n, dz') \\ &= \int_Z [T_P(f)](z') P^t(z, dz') = \int_Z f(z)' P^{t+1}(z, dz'). \end{aligned}$$

This holds for any $f \in C_b(Z, \mathbb{R})$, so by definition P^{t+1} has the Feller property, and the claim of the lemma follows by induction.

Q.E.D.

3.3.2 Monotonicity

Another important property is monotonicity. Now let Z be a subset of the euclidean m-space $\mathbb{R}^m$, and $\mathcal{Z}$ the corresponding Borel σ-algebra. A Markov kernel P on $(Z, \mathcal{Z})$ is said to be monotonic if, for any $z, z' \in Z$ such that $z \leq z'$ elementwise, $P(z', \cdot)$ first order stochastically dominates $P(z, \cdot)$; using the integral characterization of FOSD,

$$\int_Z f(z'')P(z, dz'') \geq \int_Z f(z'')P(z', dz'')$$

for any increasing function $f \in \mathcal{Z}_+$. In terms of the transformation T_P, this tells us that, given an increasing function $f \in \mathcal{Z}_+$,

$$[T_P(f)](z) = \int_Z f(z'')P(z, dz'') \geq \int_Z f(z'')P(z', dz'') = [T_P(f)](z')$$

whenever $z \leq z'$, so that $T_P(f)$ is itself an increasing function on Z. It is in this sense that T_P preserves monotonicity when the kernel P is monotonic. Heuristically, when P is monotonic, we can say that higher values of the exogenous shock today tends to lead to higher shocks tomorrow as well.

Lemma (Implications of Monotonicity)
Let P be a Markov kernel on $(Z, \mathcal{Z})$, where Z is a subset of the euclidean m-space $\mathbb{R}^m$. If P is monotonic, then the following hold true:

i) For any probability measures μ, λ on $(Z, \mathcal{Z})$ such that μ FOSD λ, the measure-kernel μP first order stochastically dominates the measure-kernel λP.

ii) For any $t \in N_+$, the product P^t is also monotonic.

Proof) Choose any increasing non-negative function $f : Z \to \mathbb{R}$, and let μ, λ be probability measures on $(Z, \mathcal{Z})$ such that μ first order stochastically dominates λ. Then,

$$\int_Z f d(\mu P) = \int_Z \int_Z f(z')P(z, dz')d\mu(z), \quad \int_Z f d(\lambda P) \ = \int_Z \int_Z f(z')P(z, dz')d\lambda(z).$$

Since the mapping

$$z \mapsto \int_Z f(z')P(z, dz')$$

is a non-negative increasing function because of the monotonicity of P, it follows from the fact that μ FOSD λ that

$$\int_Z f d(\mu P) = \int_Z \int_Z f(z') P(z, dz') d\mu(z) \geq \int_Z \int_Z f(z') P(z, dz') d\lambda(z) = \int_Z f d(\lambda P).$$

This holds for any non-negative increasing function $f : Z \to \mathbb{R}$, so μP also FOSD λP. In this sense, the monotonicity property preserves dominance relations across measure-kernel transformations as well.

For the second statement, it suffices to show that, for any $t \in N_+$ and $z, z' \in Z$ such that $z < z'$, $P^t(z', \cdot)$ first order stochastically dominates $P^t(z, \cdot)$. To this end, let $f : Z \to \mathbb{R}$ be some non-negative increasing $\mathcal{Z}$-measurable function. By monotonicity, the mapping

$$z \mapsto \int_Z f(z'') P(z, dz'')$$

is an increaisng function, and it follows immediately from the definition of monotonicity that $P(z, \cdot)$ is dominated by $P(z', \cdot)$.

Now suppose that, for some $t \geq 1$, $P^t(z', \cdot)$ FOSD $P^t(z, \cdot)$. Then, for any inon-negative increasing measurable function $f : Z \to \mathbb{R}$,

$$\begin{aligned}\int_Z f(z'') P^{t+1}(z, dz'') &= \int_Z \int_Z f(y) P(z'', dy) P^t(z, dz'') \\ &\leq \int_Z \int_Z f(y) P(z'', dy) P^t(z', dz'') = \int_Z f(z'') P^{t+1}(z', dz''),\end{aligned}$$

where the first and third equalities follow from the definition of the product of Markov kernels, and the inequality follows from the inductive hypothesis. Therefore, $P^{t+1}(z', \cdot)$ FOSD $P^{t+1}(z, \cdot)$, and our original claim follows by induction.

Q.E.D.

3.3.3 Stable Sets and the Splitting Condition

Another important concept is that of the stable set. For any $a, b \in \mathbb{R}^m$, let $[a, b]$ denote the closed m-cell

$$\prod_{i=1}^{m} [a_i, b_i] = [a_1, b_1] \times \cdots \times [a_m, b_m]$$

on $\mathbb{R}^m$. Let $(Z, \mathcal{Z})$ be a Borel subspace of the euclidean m-space, and let P be a Markov kernel on this space. A closed m-cell $[a, b]$ such that $[a, b] \subset Z$ is called a stable set for the kernel P if $P(z, [a, b]) = 1$ for any $z \in [a, b]$. In other words, once a Markov chain with kernel P is started from a point within $[a, b]$, it continues to stay in $[a, b]$. In this sense, $[a, b]$ is stable for P, and is appropriately called a stable set.

Note that $[a, b]$ is a stable set for P^t for any $t \in N_+$ if it is a stable set for P. Suppose, for some $t \geq 1$, that $P^t(z, [a, b]) = 1$ for any $z \in [a, b]$. Then, for any $z \in [a, b]$,

$$\begin{aligned}
P^{t+1}(z, [a, b]) &= \int_Z P(z', [a, b]) P^t(z, dz') \\
&= \int_{[a,b]} P(z', [a, b]) P^t(z, dz') && ([a, b] \text{ is stable for } P^t) \\
&= \int_{[a,b]} 1 P^t(z, dz') && ([a, b] \text{ is stable for } P) \\
&= P^t(z, [a, b]) = 1.
\end{aligned}$$

By induction, $P^t(z, [a, b]) = 1$ for any $t \in N_+$.

We introduce the splitting condition as a technical assumption that will be needed later to ensure the weak convergence of Markov chains. Suppose $(Z, \mathcal{Z})$ is a Borel subspace of the euclidean m-space, and let $[a, b]$ be a closed m-cell that is a subset of Z. A Markov kernel P on $(Z, \mathcal{Z})$ is said to satisfy the splitting condition on $[a, b]$ if there exists some $c \in [a, b]$, $\epsilon > 0$ and $N \in N_+$ such that

$$P^N(a, [c, b]) \geq \epsilon, \quad P^N(b, [a, c]) \geq \epsilon.$$

In other words, even if our initial values are given as the most extreme values in the m-cell $[a, b]$, it must be possible to reach across some "midpoint" c in $[a, b]$ after a finite number of iterations. In this specification, the space $[a, b]$ is split with c in the middle, hence the name for this condition. We can think of this as a weak version of the irreducibility

condition that is often required for the convergence of Markov chains in total variation distance.

Below is a result that formalizes this intuitive understanding of the splitting condition.

Lemma (Implications of the Splitting Condition)
Let P be a Markov kernel on $(Z, \mathcal{Z})$, where $(Z, \mathcal{Z})$ is a Borel subspace of the euclidean m-space that contains the closed m-cell $[a, b]$. Suppose P is monotonic and satisfies the splitting condition on $[a, b]$ for some $c \in [a, b]$, $\epsilon > 0$ and $N \in N_+$. Then, the following hold true:

i) For any $z \in [a, b]$, $P^N(z, [a, c]) \geq \epsilon$ and $P^N(z, [c, b]) \geq \epsilon$.

ii) For any bounded and increasing $\mathcal{Z}$-measurable function $f : Z \to \mathbb{R}$,

$$\epsilon f(c) + (1 - \epsilon) f(a) \leq \int_Z f(z') P^N(z, dz') \leq \epsilon f(c) + (1 - \epsilon) f(b)$$

for any $z \in [a, b]$.

Proof) i) We first note that, for any $t \in N_+$, P^t is monotonic by the directly preceding result. Furthermore, since $[a, b]$ is a stable set for P, it is also stable for P^t.

Now note that $f = I_{[c,+\infty) \cap Z}$ is a non-negative increasing measurable function on Z, and that, for any $z \in [a, b]$, $a \leq z$. For any $t \in N_+$, since $[a, b]$ is stable for P^t,

$$P^t(z, [c, +\infty) \cap Z) = P^t(z, [c, +\infty) \cap [a, b]) = P^t(z, [c, b])$$

for any $z \in [a, b]$. Then, by the monotonicity of P^t,

$$\begin{aligned} P^t(z, [c, b]) = P^t(z, [c, +\infty) \cap Z) &= \int_Z f(z'') P^t(z, dz'') \\ &\geq \int_Z f(z'') P^t(a, dz'') = P^t(a, [c, +\infty) \cap Z) = P^t(a, [c, b]). \end{aligned}$$

Putting $t = N$ yields

$$P^N(z, [c, b]) \geq P^N(a, [c, b]) \geq \epsilon.$$

Likewise, define $g = I_{(-\infty, c] \cap Z}$. Then, g is a non-negative decreasing measurable function on Z such that

$$\int_Z g(z') P^t(z, dz') = P^t(z, (-\infty, c] \cap Z) = P^t(z, (-\infty, c] \cap [a, b]) = P^t(z, [a, c])$$

for any $z \in [a, b]$ and $t \in N_+$. As such, for any $z \in [a, b]$, since $P^N(b, \cdot)$ FOSD $P^N(z, \cdot)$, we have

$$P^N(z, [a, c]) = \int_Z g(z') P^N(z, dz') \geq \int_Z g(z') P^N(b, dz') = P^N(b, [a, c]).$$

It follows that

$$P^N(z, [a, c]) \geq P^N(b, [a, c]) \geq \epsilon.$$

ii) Let f be a bounded and increasing $\mathcal{Z}$-measurable function. Then, for any $z \in [a, b]$, since $[a, b]$ is stable for P^N, we have

$$\begin{aligned}
\int_Z f(z') P^N(z, dz') &= \int_{[a,b]} f(z') P^N(z, dz') \\
&= \int_{[a,c]} f(z') P^N(z, dz') + \int_{[a,b] \setminus [a,c]} f(z') P^N(z, dz') \\
&\leq f(c) \cdot P^N(z, [a, c]) + f(b) \cdot P^N(z, [a, b] \setminus [a, c]),
\end{aligned}$$

where the inequality follows from the fact that f is increasing and thus $f(c) = \mathsf{sup}_{x \in [a,c]} f(x)$ and $f(b) = \mathsf{sup}_{x \in [a,b] \setminus [a,c]} f(x)$. Since $P^N(z, [a, c]) \geq \epsilon$, the right hand side becomes

$$f(c) \cdot P^N(z, [a, c]) + f(b) \cdot P^N(z, [a, b] \setminus [a, c]) = f(c) \cdot P^N(z, [a, c]) + f(b) \cdot \Big(1 - P^N(z, [a, c])\Big)$$

$$= (f(c) - f(b)) \cdot P^N(z, [a, c]) + f(b)$$

$$\leq \epsilon(f(c) - f(b)) + f(b) = \epsilon f(c) + (1 - \epsilon) f(b),$$

where the inequality follows from the fact that $f(c) - f(b) \leq 0$.

Similarly,

$$\int_Z f(z') P^N(z, dz') = \int_{[a,b]} f(z') P^N(z, dz')$$

$$\int_{[c,b]} f(z') P^N(z, dz') + \int_{[a,b] \setminus [c,b]} f(z') P^N(z, dz')$$

$$\geq f(c) \cdot P^N(z, [c, b]) + f(a) \cdot P^N(z, [a, b] \setminus [c, b])$$

$$= (f(c) - f(a)) P^N(z, [c, b]) + f(a)$$

$$\geq (f(c) - f(a)) \epsilon + f(a) = \epsilon f(c) + (1 - \epsilon) f(a),$$

where each step is justified by the same logic as above.

Q.E.D.

The first result above helps with the heuristic interpretation of the splitting condition. Together with monotonicity, it allows the Markov chain to reach any point in the stable set $[a, b]$ with positive probability starting from any point z in that set, given a sufficient number of iterations. The second result then shows us that the integral of any increasing function can be bound above and below by a convex combination of the function values of the endpoints and the "midpoint" c.

3.4 Markov Chains

So far, we have focused only on Markov kernels. Here we show how they can be applied to the actual stochastic processes of Markov chains. The exposition below can be equivalently formulated solely in terms of transition probability kernels, but we work with random variables instead of distributions to make the concepts clearer and (hopefully) more intuitive.

Letting $(\Omega, \mathcal{H}, \mathbb{P})$ be some underlying probability space, we say that the stochastic process $\{Y_t\}_{t\in\mathbb{N}}$ with state space $(Z, \mathcal{Z})$ is a Markov chain with transition probability P if P is a Markov kernel on $(Z, \mathcal{Z})$ such that, for any $t \in \mathbb{N}$,

$$\mathbb{P}(Y_{t+1} \in A \mid Y_t, \cdots, Y_0) = \mathbb{P}(Y_{t+1} \in A \mid Y_t) = P(Y_t, A)$$

for any $A \in \mathcal{Z}$. Equivalently,

$$\mathbb{E}[f \circ Y_{t+1} \mid Y_t, \cdots, Y_0] = \mathbb{P}(f \circ Y_{t+1} \mid Y_t) = \int_Z f(z) P(Y_t, dz) = T_P(f) \circ Y_t$$

for any $\mathcal{Z}$-measurable real function f. Heuristically, $\{Y_t\}_{t\in\mathbb{N}}$ is a Markov chain if the conditional distribution of Y_{t+1} given all past values $Y_t, \cdots, Y_1$ depends only on the value Y_t in the immediately preceding period.

3.4.1 Distributions of Markov Chains

Suppose the initial distribution of the chain is μ. Then, the distribution of Y_1 is given as

$$\mathbb{P}(Y_1 \in A) = \mathbb{E}[\mathbb{P}(Y_1 \in A \mid Y_0)] = \int_Z P(z_0, A) d\mu(z_0)$$

for any $A \in \mathcal{Z}$, so that the measure-kernel μP is the distribution of Y_1. Suppose, for some $t \geq 1$, that we have shown the measure μP^t defined as

$$(\mu P^t)(A) = \int_Z P^t(z_0, A) d\mu(z_0) \quad \text{for any } A \in \mathcal{Z}$$

is the distribution of Y_t. Then,

$$\mathbb{P}(Y_{t+1} \in A) = \mathbb{E}[\mathbb{P}(Y_{t+1} \mid A \mid Y_t)] = \int_Z P(z, A) d(\mu P^t)(z)$$

$$= \int_Z [T_{P^t}(P(\cdot, A))](z_0) d\mu(z_0) = \int_Z \left[\int_Z P(z, A) P^t(z_0, dz) \right] d\mu(z_0)$$

$$= \int_Z P^{t+1}(z_0, A) d\mu(z_0)$$

for any $A \in \mathcal{Z}$, so that μP^{t+1} is the distribution of Y_{t+1}. By induction, the distribution of each Y_t is the measure-kernel μP^t. We can also see by a similar process that the condition distribution of Y_t given $Y_0 = z_0$ is the Markov kernel P^t, for any $t \in N_+$.

3.4.2 Joint Distribution of Elements in a Markov Chain

A remarkable feature of Markov chains is that, even when $(Z, \mathcal{Z})$ is not a standard measurable space, there exist regular versions of the conditional distribution of $Y_1, \cdots, Y_t$ given the initial value Y_0 for any period $t \in N_+$, and furthermore, we can recover these regular versions recursively. This is the content of the following lemma:

Lemma (Distributions of Markov Chains)
Let $\{Y_t\}_{t \in \mathbb{N}}$ be a Markov chain with state space $(Z, \mathcal{Z})$ and transition probability P. Let $\{\mu^t\}_{t \in N_+}$ be a sequence of functions such that $\mu^1 = P$ and, for each $t \geq 2$, $\mu^t : Z \times \mathcal{Z}^t \to [0, 1]$ is a transition probability kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^t$ defined recursively as

$$\mu^t(z_0, A) = \int_{Z^{t-1}} \int_Z I_A(z^{t-1}, z_t) P(z_{t-1}, dz_t) \mu^{t-1}(z_0, dz^{t-1})$$

for any $(z_0, A) \in Z \times \mathcal{Z}^t$. Then,

$$\mathbb{E}\left[f \circ Y^t \mid Y_0 \right] = \int_{Z^t} f(z^t) \mu^t(Y_0, dz^t)$$

$$:= \int_{Z^{t-1}} \int_Z f(z^{t-1}, z_t) P(z_{t-1}, dz_t) \mu^{t-1}(Y_0, dz^{t-1})$$

for any $\mathcal{Z}^t$-measurable non-negative or bounded function f.

Proof) $\mu^1 = P$ is clearly a transition probability kernel from $(Z, \mathcal{Z})$ into itself, and

$$\mathbb{P}(Y_1 \in A \mid Y_0) = P(Y_0, A)$$

for any $A \in \mathcal{Z}$ by definition, so that, by implication,

$$\mathbb{E}[f \circ Y_1 \mid Y_0] = \int_Z f(z_1) P(Y_0, dz_1) = \int_Z f(z_1) \mu^1(Y_0, dz_1)$$

for any $f \in \mathcal{Z}_{+,b}$.

Now suppose that, for some $t \geq 1$, μ^t has been shown to be a transition probability kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^t$ such that

$$\mathbb{E}\left[f \circ Y^t \mid Y_0\right] = \int_{Z^t} f(z^t) \mu^t(Y_0, dz^t)$$

for any $f \in (\mathcal{Z}^t)_{+,b}$. We first check that μ^{t+1} is a transition probability kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^t$. Fixing $z_0 \in Z$,

$$\mu^{t+1}(z_0, A) = \int_{Z^t} [T_P(I_A)](z^t) \mu^t(z_0, dz^t)$$

for any $A \in \mathcal{Z}^{t+1} = \mathcal{Z}^t \bigotimes \mathcal{Z}$, where $T_P(I_A)$ is a non-negative $\mathcal{Z}^t$-measurable function and $\mu^t(z_0, \cdot)$ is a probability measure on $(Z, \mathcal{Z}^t)$, so $\mu^{t+1}(z_0, \cdot)$ is a probability measure on $(Z, \mathcal{Z})^t \bigotimes (Z, \mathcal{Z}) = (Z, \mathcal{Z})^{t+1}$. On the other hand, fixing $A \in \mathcal{Z}^{t+1}$,

$$\mu^{t+1}(\cdot, A) = T_{\mu^t}(T_P I_A),$$

where μ^t is a transition probability kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^t$ and $T_P(I_A) \in (\mathcal{Z}^t)_+$, we can see that $\mu^{t+1}(\cdot, A)$ is a non-negative $\mathcal{Z}$-measurable function. This proves that μ^{t+1} is a transition probabiltiy kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^{t+1}$.

For any $A \in \mathcal{Z}^t$ and $B \in \mathcal{Z}$, by definition

$$\begin{aligned} \mu^{t+1}(z_0, A \times B) &= \int_{Z^t} \int_Z I_{A \times B}(z^t, z_{t+1}) P(z_t, dz_{t+1}) \mu^t(z_0, dz^t) \\ &= \int_A P(z_t, B) \mu^t(z_0, dz^t). \end{aligned}$$

Define $f : Z^t \to \mathbb{R}_+$ as

$$f(z^t) = P(z_t, B) \cdot I_A(z^t)$$

for any $z^t = (z^{t-1}, z_t) \in Z^t$; then, f is a non-negative $\mathcal{Z}^t$-measurable function, so that, by the inductive hypothesis,

$$\begin{aligned}
\mu^{t+1}(\cdot, A \times B) \circ Y_0 &= \int_{Z^t} f(z^t) \mu^t(Y_0, dz^t) = \mathbb{E}\left[f \circ Y^t \mid Y_0\right] \\
&= \mathbb{E}\left[P(Y_t, B) \cdot (I_A \circ Y^t) \mid Y_0\right] \\
&= \mathbb{E}\left[\mathbb{P}(Y_{t+1} \in B \mid Y_t) \cdot (I_A \circ Y^t) \mid Y_0\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(Y_{t+1} \in B \mid Y^t\right) \cdot (I_A \circ Y^t) \mid Y_0\right] \quad \text{(Markov Property)} \\
&= \mathbb{E}\left[\mathbb{E}\left[I_{\{Y_{t+1} \in B\} \cap \{Y^t \in A\}} \mid Y^t\right] \mid Y_0\right] \\
& \qquad\qquad \text{(Conditional Determinism)} \\
&= \mathbb{E}\left[I_{\{Y_{t+1} \in B\} \cap \{Y^t \in A\}} \mid Y_0\right] \qquad \text{(Tower Property)} \\
&= \mathbb{P}\left(Y^t \in A, Y_{t+1} \in B \mid Y_0\right).
\end{aligned}$$

Since the collection $A \times B$ of measurable rectangles on $Z^t \times Z = Z^{t+1}$ is a π-system generating the product σ-algebra $\mathcal{Z}^{t+1}$, the $\pi - \lambda$ theorem reveals that

$$\mu^{t+1}(\cdot, A) \circ Y_0 = \mathbb{P}\left(Y^{t+1} \in A \mid Y_0\right)$$

for any $A \in \mathcal{Z}^{t+1}$. This shows us that the mapping $(\omega, A) \mapsto \mu^{t+1}(Y_0(\omega), A)$ is a regular version of the conditional distribution of Y^{t+1} given Y_0, and therefore that

$$\mathbb{E}\left[f \circ Y^{t+1} \mid Y_0\right] = \int_{Z^{t+1}} f(z^{t+1}) \mu^{t+1}(Y_0, dz_{t+1})$$

for any $f \in (\mathcal{Z}^{t+1})_{+,b}$.

The claim now follows by induction.

Q.E.D.

We have just shown that, for any $t \in N_+$, $\mu^t(z_0, \cdot)$ is the conditional distribution of $Y_1, \cdots, Y_t$ given $Y_0 = z_0$. Since μ^t is defined using only the transition probability P,

it shows us that knowledge of the transition probability P (alongside, technically, the distribution of the initial value Y_0) is sufficient to characterize the probability law of any Markov chain $\{Y_t\}_{t\in\mathbb{N}}$.

3.4.3 Stationary Distributions for Markov Chains

So far, we have not mentioned the distribution of the initial value Y_0 of the Markov chain at all. There are many ways to specify this initial distribution, but here we choose to assume that the initial distribution is a stationary distribution of the relevant Markov kernel P.

Given a Markov kernel P on $(Z, \mathcal{Z})$, μ_0 is a stationary distribution for P if

$$\int_Z P(z, A) d\mu_0(z) = \mu_0(A)$$

for any $A \in \mathcal{Z}$. To elucidate the sense in which μ_0 is "stationary" for P, consider a Markov chain $\{Y_t\}_{t\in\mathbb{N}}$ with transition probability P. Suppose the distribution of Y_0 is μ_0. Then,

$$\begin{aligned}\mathbb{P}(Y_1 \in A) = \mathbb{E}[\mathbb{P}(Y_1 \in A \mid Y_0)] &= \mathbb{E}[P(Y_0, A)] \\ &= \int_Z P(z_0, A) d\mu_0(z_0) = \mu_0(A)\end{aligned}$$

for any $A \in \mathcal{Z}$, so that μ_0 is the (unconditional) distribution of Y_1. Suppose that, for some $t \geq 1$, the distribution of Y_t is μ_0. Then,

$$\begin{aligned}\mathbb{P}(Y_{t+1} \in A) = \mathbb{E}[\mathbb{P}(Y_{t+1} \in A \mid Y_t, \cdots, Y_0)] &= \mathbb{E}[P(Y_t, A)] \\ &= \int_Z P(z_t, A) d\mu_0(z_t) = \mu_0(A)\end{aligned}$$

for any $A \in \mathcal{Z}$, showing us that the distribution of Y_{t+1} is also μ_0.

Therefore, if the initial distribution is μ_0, then the distribution of all subsequent values in the chain is also μ_0. In other words, given the initial distribution μ_0, the chain is stationary at μ_0. A Markov chain with transition probability P whose initial distribution is μ_0, or which is started from a stationary distribution of P, is called a stationary Markov chain.

3.5 Weak Convergence of Markov Chains

Let Z be a subset of the euclidean m-space $\mathbb{R}^m$, and $\mathcal{Z}$ the corresponding Borel σ-algebra. Consider a Markov kernel P on $(Z, \mathcal{Z})$. In many cases, we want to find a stationary distribution μ_0 to which a Markov chain with kernel P converges to weakly, regardless of the choice of initial distribution; this is the case, for instance, when running Markov Chain Monte Carlo (MCMC) algorithms, where we want to sample from a target distribution by running a Markov chain that converges to that distribution. In this section we investigate some sufficient conditions under which P not only admits a stationary distribution, but converges to it regardless of the choice of the initial distribution. It turns out that the three properties we studied above, namely the Feller property, monotonicity and the splitting condition, are sufficient for the result we desire.

Theorem (Existence of Stationary Distribution under Feller Property) Let P be a Markov kernel on $(Z, \mathcal{Z})$, where (Z, ρ) is a complete and separable metric space. Suppose there exists an initial distribution μ_0 on $(Z, \mathcal{Z})$ such that the sequence $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ of measure-kernels converges weakly to some probability measure μ on $(Z, \mathcal{Z})$. If P possesses the Feller property, then μ is stationary for P, that is, $\mu P = \mu$.

Proof) Choose any probability measure μ_0 on $(Z, \mathcal{Z})$, and consider the sequence $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ of probability measures on $(Z, \mathcal{Z})$. Suppose that $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ converges weakly to some μ^*. We want to show that μ^* is stationary for P.

Define the sequence $\{\mu_n\}_{n\in N_+}$ as

$$\mu_n = \frac{1}{n}\sum_{t=0}^{n-1} \mu_0 P^t$$

for any $n \in N_+$; since the non-negative linear combination of measures is also a measure, and the total mass of each μ_n is 1, $\{\mu_n\}_{n\in N_+}$ is a sequence of probability measures on $(Z, \mathcal{Z})$. For any $A \in \mathcal{Z}$ such that $\mu^*(\partial A)$, the sequence $\{\mu_0 P^t(A)\}_{t\in\mathbb{N}}$ converges to $\mu^*(A)$ by the Portmanteau theorem; since the convergence of a sequence implies convergence of the arithmetic mean, this implies that

$$\lim_{n\to\infty} \mu_n(A) = \mu^*(A).$$

This holds for any $A \in \mathcal{Z}$ such that $\mu^*(\partial A)$, so by the Portmanteau theorem again, $\{\mu_n\}_{n\in N_+}$ converges weakly to μ^*.

Choose any continuous and bounded function $f \in C_b(Z, \mathbb{R})$. $T_P(f)$ is also continuous and bounded by the Feller property, so that, by the definition of weak convergence,

$$\lim_{n\to\infty} \int_Z f d\mu_n = \int_Z f d\mu^*$$

$$\lim_{n\to\infty} \int_Z T_P(f) d\mu_n = \int_Z T_P(f) d\mu^*.$$

Let $\epsilon > 0$. The above limit results tell us that there exists some $N \in N_+$ such that

$$\left| \int_Z f d\mu_n - \int_Z f d\mu^* \right| < \frac{\epsilon}{3}, \quad \text{and} \quad \left| \int_Z T_P(f) d\mu_n - \int_Z T_P(f) d\mu^* \right| < \frac{\epsilon}{3}$$

for any $n \geq N$.

Now choose any $n \in N_+$, and note that

$$\left| \int_Z f d\mu_n - \int_Z T_P(f) d\mu_n \right| = \left| \frac{1}{n} \sum_{t=0}^{n-1} \left[\int_Z f d(\mu_0 P^{T_i}) - \int_Z T_P(f) d(\mu_0 P^{T_i}) \right] \right|.$$

By definition of measure-kernels, we can see that

$$\int_Z f d(\mu_0 P^t) = \int_Z T_{P^t}(f) d\mu_0 = \int_Z \int_Z f(z') P^t(z, dz') d\mu_0(z)$$

and

$$\begin{aligned}
\int_Z T_P(f) d(\mu_0 P^t) &= \int_Z \int_Z [T_p(f)](z') P^t(z, dz') d\mu_0(z) \\
&= \int_Z \left[\int_Z \int_Z f(z'') P(z', dz'') P^t(z, dz') \right] d\mu_0(z) \\
&= \int_Z \int_Z f(z') P^{t+1}(z, dz') d\mu_0(z) = \int_Z f d(\mu_0 P^{t+1})
\end{aligned}$$

for any $0 \leq t \leq T-1$. It follows that

$$\begin{aligned}\left|\int_Z f d\mu_n - \int_Z T_P(f) d\mu_n\right| &= \left|\frac{1}{n}\int_Z f d\mu_0 - \frac{1}{n}\int_Z f d(\mu_0 P^n)\right| \\ &\leq \frac{1}{n}\int_Z |f| d\mu_0 + \frac{1}{n}\int_Z |f| d(\mu_0 P^n) \\ &\leq \frac{2}{n}\|f\|_\infty,\end{aligned}$$

where $\|\cdot\|_\infty$ is the supremum norm on $C_b(Z, \mathbb{R})$. Thus,

$$\lim_{n\to\infty}\left|\int_Z f d\mu_n - \int_Z T_P(f) d\mu_n\right| = 0.$$

This indicates that we we can choose $N \in N_+$ large enough so that

$$\left|\int_Z f d\mu_n - \int_Z T_P(f) d\mu_n\right| < \frac{\epsilon}{3}$$

for any $n \geq N$.

Putting all these results together, we can see that, if we put $n = N$, then

$$\begin{aligned}\left|\int_Z f d\mu^* - \int_Z T_P(f) d\mu^*\right| &\leq \left|\int_Z f d\mu_n - \int_Z f d\mu^*\right| + \left|\int_Z f d\mu_n - \int_Z T_P(f) d\mu_n\right| \\ &\quad + \left|\int_Z T_P(f) d\mu_n - \int_Z T_P(f) d\mu^*\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

This holds for any $\epsilon > 0$, so

$$\int_Z f d\mu^* = \int_Z T_P(f) d\mu^* = \int_Z f d(\mu^* P).$$

μ^* and $\mu^* P$ are probability measures $(Z, \mathcal{Z})$ whose integrals agree for continuous and bounded functions $f \in C_b(Z, \mathbb{R})$. It follows from the uniqueness of weak limits[1] that $\mu^* = \mu^* P$ on $(Z, \mathcal{Z})$, which establishes that μ^* is a stationary distribution of P.

[1]For reference, check out lemma 4.2 of the probability theory text.

Q.E.D.

With the Feller property, we are able to establish that the weak limit of Markov chains is a stationary distribution. It is in itself a remarkable result, but it shines when combined with the monotonicity property. Given a Markov kernel P with both monotonicity and the Feller property, we can show not only that a stationary distribution exists for P, but that any Markov chain with kernel P converges weakly to some stationary distribution of P when started from a particular initial distribution.

Theorem (Weak Convergence to Stationary Distribution)
Let P be a Markov kernel on $(Z, \mathcal{Z})$, where $(Z, \mathcal{Z})$ is a Borel subspace of the euclidean m-space $\mathbb{R}^m$ that contains the closed m-cell $[a, b]$. Suppose that:

i) P is monotonic and possesses the Feller property.

ii) $[a, b]$ is a stable set for P, that is, $P(z, [a, b]) = 1$ for any $z \in [a, b]$.

Then, any Markov chain $\{Y_t\}_{t\in\mathbb{N}}$ with kernel P and initial distribution $\mu_0 = \delta_a$ or δ_b converges weakly to some stationary distribution μ^* of P.

Proof) Consider a Markov chain $\{Y_t\}_{t\in\mathbb{N}}$ with initial distribution μ_0 and kernel P. We saw above that the corresponding sequence of distributions is given by the measure-kernels $\{\mu_0 P^t\}_{t\in\mathbb{N}}$. We first consider the case where $\mu_0 = \delta_a$. The idea of the proof is as follows. Since P is monotonic, it preserves first order stochastic dominance across measure-kernel transformations, as we saw earlier. It will then follow that the sequence $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ is increasing in the sense that $\mu_0 P^{t+1}$ FOSD $\mu_0 P^t$ for any $t \in \mathbb{N}$. Using this, we can show that $\{\mu_0 P^t(A)\}_{t\in\mathbb{N}}$ converges monotonically to some limit in $[0, 1]$ for any $A \in \mathcal{Z}$ that belongs to some π-system generating $\mathcal{Z}$. This will be sufficient to show that any weakly convergent subsequence of $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ converges to the same limit; and combined with Prohorov's theorem, we will be able to conclude that $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ converges weakly to some probability measure on $(Z, \mathcal{Z})$. Finally, the Feller property of P is used to show that this weak limit is stationary for P.

Step 1: Dominance Relations and Monotonicity

Suppose $\mu_0 = \delta_a$. Then, for any non-negative increasing $\mathcal{Z}$-measurable function $f : Z \to \mathbb{R}$,

$$\int_Z f d\mu_0 = f(a).$$

Meanwhile, because $P(a, [a, b]) = 1$, we have

$$\int_Z f d(\mu_0 P) = \int_Z f(z) P(a, dz') = \int_{[a,b]} f(z) P(a, dz') \geq f(a) = \int_Z f d\mu_0,$$

where we used the fact that f, being an increasing function, satisfies $f(a) \leq f(z)$ for any $z \in [a, b]$. It follows that $\mu_0 P$ first order stochastically dominates μ_0. Suppose that, for some $t \geq 1$, $\mu_0 P^t$ FOSD $\mu_0 P^{t-1}$. Then, by the preservation of dominance across measure-kernel transformations,

$$\mu_0 P^{t+1} = (\mu_0 P^t) P \quad \text{FOSD} \quad \mu_0 P^t = (\mu_0 P^{t-1}) P.$$

It follows that $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ is a sequence of probability measures on $(Z, \mathcal{Z})$ that become more dominant as time passes.

Recall that the collection $\mathbb{B} = \{(x, +\infty) \cap Z \mid x \in \mathbb{R}^m\}$ is a π-system that generates the Borel σ-algebra $\mathcal{Z}$, where $(x, +\infty)$ is taken to be the product of rays

$$(x, +\infty) = \prod_{i=1}^{m} (x_i, +\infty) \subset \mathbb{R}^m.$$

For any $B \in \mathbb{B}$, since $f = I_B$ is an non-negative increasing and measurable function, we can see that

$$(\mu_0 P^t)(B) = \int_Z f(z) d(\mu_0 P^t)(z) \leq \int_Z f(z) d(\mu_0 P^{t+1})(z) = (\mu_0 P^{t+1})(B)$$

for any $t \in \mathbb{N}$. Therefore, $\{(\mu_0 P^t)(B)\}_{t \in \mathbb{N}}$ is a monotonically increasing real sequence that is bounded above by 1; it follows that it converges to a limit $L_B \in [0, 1]$.

Step 2: Existence of a Weak Limit

We are now all ready to prove the convergence result. First, we show that any weakly convergent subsequence of $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ converges to the same limit. To see this, let $\{\mu_0 P^{T_n}\}_{n\in N_+}$ and $\{\mu_0 P^{T_k}\}_{k\in N_+}$ be two subsequences of $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ that converge weakly to the probability measures $\tilde{\mu}$ and $\tilde{v}$ on $(Z,\mathcal{Z})$. Since $\tilde{\mu}$ and $\tilde{v}$ are probability measures, there are at most countably many $x\in\mathbb{R}^m$ such that $\tilde{\mu}(\partial[(x,+\infty)\cap Z])>0$ and $\tilde{v}(\partial[(x,+\infty)\cap Z])>0$; let these xs be collected in the set Z_0. Then, the set $\mathbb{B}'=\{(x,+\infty)\cap Z \mid x\in Z_0^c\}$ is a π-system generating $\mathcal{Z}$[2]. For any $B\in\mathbb{B}'$, by the Portmanteau theorem

$$\lim_{n\to\infty}\mu_0 P^{T_n}(B)=\tilde{\mu}(B) \quad \text{and} \quad \lim_{k\to\infty}\mu_0 P^{T_k}(B)=\tilde{v}(B).$$

But recall that $\{(\mu_0 P^t)(B)\}_{t\in\mathbb{N}}$ is an increasing sequence with limit L_B; therefore, we can conclude that

$$\tilde{\mu}(B)=L_B=\tilde{v}(B).$$

This holds for any $B\in\mathbb{B}'$, so that $\tilde{\mu}=\tilde{v}$ on a the π-system $\mathbb{B}'$ generating $\mathcal{Z}$. $\tilde{\mu}$ and $\tilde{v}$ are both finite measures, so this implies that $\tilde{\mu}=\tilde{v}$ on $(Z,\mathcal{Z})$. Let μ^* denote this common weak limit.

Now choose any subsequence $\{\mu_0 P^{T_n}\}_{n\in N_+}$ of $\{\mu_0 P^t\}_{t\in\mathbb{N}}$. Since

$$(\mu_0 P^{T_n})([a,b])=1$$

for any $n\in N_+$, and $[a,b]$ is a compact subset of Z, $\{\mu_0 P^{T_n}\}_{n\in N_+}$ is a tight sequence of probability measures on $(Z,\mathcal{Z})$. Thus, by Prohorov's theorem, there exists a further subsequence of $\{\mu_0 P^{T_n}\}_{n\in N_+}$ that converges weakly to some probability measure on $(Z,\mathcal{Z})$. By what we just proved above, this limit is μ^*. It follows from the selection theorem for subsequences that $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ itself converges weakly to μ^*. The preceding theorem now tells us that μ^* is a stationary distribution of P.

[2]For a rigorous proof of this result, consult lemma 1.9 of the probability theory text.

The Case $\mu_0 = \delta_b$

For the case where $\mu_0 = \delta_b$, we can proceed in much the same way by noting that this time, $\mu_0 P^t$ FOSD $\mu_0 P^{t+1}$ for any $t \in \mathbb{N}$. To see this, we initially observe that, for any non-negative increasing measurable function f, since $f(b) \geq f(z)$ for any $z \in Z$,

$$\int_Z f d\mu_0 = f(b) \geq \int_Z f d(\mu_0 P).$$

This shows us that μ_0 first order stochastically dominates $\mu_0 P$, and from there on we can proced by repeatedly applying the fact that P preserves dominance relations across measure-kernel transformations because it is monotonic. Once we have established this about $\{\mu_0 P^t\}_{t\in\mathbb{N}}$, we can use the fact that, for any $B \in \mathbb{B} = \{(x, +\infty) \cap Z \mid x \in \mathbb{R}^m\}$, $\{\mu_0 P^t(B)\}_{t\in\mathbb{N}}$ is a monotonically decreasing sequence bounded below by 0 to conclude that it convergs to some $U_B \in [0, 1]$. After that, the proof is identical to the above.

Q.E.D.

While the Feller property and monotonicity are sufficient to establish that a Markov chain converges weakly given extreme initial distributions δ_a and δ_b, it is nevertheless insufficient when we want to assert that there exists a unique stationary distribution of P to which any Markov chain converges weakly, regardless of the initial distribution. This is where the splitting condition comes in; by allowing virtually any state to be reached starting from any initial state given a sufficient number of iterations, the splitting condition helps the Markov chain in question move all over the state space Z and thus converge to a stationary distribution, no matter where it starts.

Theorem (Weak Convergence to a Unique Stationary Distribution)
Let P be a Markov kernel on $(Z, \mathcal{Z})$, where $(Z, \mathcal{Z})$ is a Borel subspace of the euclidean m-space that contains the closed m-cell $[a, b]$. Suppose that:

i) P possesses the Feller property and is monotonic.

ii) $[a, b]$ is a stable set of P on which it satisfies the splitting condition for some $c \in [a, b]$, $\epsilon > 0$ and $N \in N_+$.

iii) A Markov chain with kernel P eventually enters $[a, b]$, that is, for any probability measure μ_0 on $(Z, \mathcal{Z})$,

$$\lim_{T \to \infty} (\mu_0 P^T)([a, b]) = 1.$$

Then, there exists a stationary distribution μ^* of P that assigns all its mass to $[a, b]$ such that any Markov chain $\{Y_t\}_{t \in \mathbb{N}}$ with kernel P converges in distribution to μ^*.

In addition, P admits a unique stationary distribution, that is, if μ is a stationary distribution of P, then $\mu = \mu^*$ on $(Z, \mathcal{Z})$.

Proof) **Case 1: Initial Distribution on Stable Set**

We first consider the case where the initial distribution μ_0 assigns mass 1 to the stable set $[a, b]$. Consider the sequence $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ of probability measures on $(Z, \mathcal{Z})$. By the compactness of the subset $[a, b]$ of Z and $(\mu_0 P^t)([a, b]) = 1$ for any $t \in \mathbb{N}$ in this case, $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ is a tight sequence of probability measures and therefore any subsequence of $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ admits a further subsequence that converges weakly to some probability measure μ on $(Z, \mathcal{Z})$. The main content of our proof is that this μ is the same for every weakly convergent subsequence of $\{\mu_0 P^t\}_{t \in \mathbb{N}}$, which establishes that $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ converges weakly to some probability measure. The Feller property then shows us that this measure is stationary for P.

Let μ be the weak limit of some weakly convergent subsequence of $\{\mu_0 P^t\}_{t \in \mathbb{N}}$;

let this subsequence be $\{\mu_0 P^{T_n}\}_{t\in\mathbb{N}}$ itself. Since μ_0 is concentrated on $[a,b]$,

$$\delta_b \quad \text{FOSD} \quad \mu_0 \quad \text{FOSD} \quad \delta_a.$$

P is monotonic and preserves dominance relations, so by implication,

$$\delta_b P^t \quad \text{FOSD} \quad \mu_0 P^t \quad \text{FOSD} \quad \delta_a P^t$$

for any $t \in \mathbb{N}$. Furthermore, since $[a,b]$ is a stable set for P, $\{\delta_a P^t\}_{t\in\mathbb{N}}$ and $\{\delta_b P^t\}_{t\in\mathbb{N}}$ converge weakly to some stationary distributions μ_a^* and μ_b^* of P. In particular, for any non-negative increasing function $f : Z \to \mathbb{R}$,

$$\int_Z f d(\delta_a P^{T_n}) \leq \int_Z f d(\mu_0 P^{T_n}) \leq \int_Z f d(\delta_b P^{T_n})$$

for any $n \geq N_0$. Let $\mathbb{B} = \{(x,+\infty)\cap Z \mid x \in \mathbb{R}^m\}$; this is a π-system that generates $\mathcal{Z}$, and define

$$\mathbb{B}' = \{B \in \mathbb{B} \mid \mu_a^*(\partial B) = \mu_b^*(\partial B) = \mu(\partial B) = 0\}.$$

Then, $\mathbb{B}'$ is also a π-system that generates $\mathcal{Z}$, $f = I_B$ is a non-negative increasing function for any $B \in \mathbb{B}'$, and by the Portmanteau theorem,

$$\lim_{n\to\infty} \int_Z f d(\delta_i P^{T_n}) = \mu_i^*(B), \quad \lim_{n\to\infty} \int_Z f d(\mu_0 P^{T_n}) = \mu(B)$$

for $i = a, b$. As such, taking limits on the above inequality with $f = I_B$ shows us that

$$\mu_a^*(B) \leq \mu(B) \leq \mu_b^*(B)$$

for any $B \in \mathbb{B}'$.

Fixing $B \in \mathbb{B}'$, we now show that $\mu_a^*(B) = \mu_b^*(B)$. By our choice of the set B,

$$\lim_{t\to\infty} (\delta_a P^t)(B) = \mu_a^*(B), \quad \lim_{t\to\infty} (\delta_b P^t)(B) = \mu_b^*(B)$$

by the Portmanteau theorem. Since $\{N \cdot t\}_{t \in \mathbb{N}}$ is a subsequence of $\mathbb{N}$ and

$$(\delta_a P^t)(B) = P^t(a, B), \quad (\delta_b P^t)(B) = P^t(b, B)$$

for any $t \in \mathbb{N}$, it follows that

$$\lim_{t \to \infty} P^{Nt}(a, B) = \mu_a^*(B), \quad \lim_{t \to \infty} P^{Nt}(b, B) = \mu_b^*(B).$$

Since $P^{N(t-1)}(\cdot, B)$ is a bounded and increasing function on Z by monotonicity and the choice of B, and $[a, b]$ is a stable set for P, it follows from the implications of the splitting condition that

$$\begin{aligned} \epsilon \cdot P^{N(t-1)}(c, B) + (1-\epsilon) P^{N(t-1)}(a, B) &\leq \int_Z P^{N(t-1)}(z', B) P^N(z, dz') = P^{Nt}(z, B) \\ &\leq \epsilon P^{N(t-1)}(c, B) + (1-\epsilon) P^{N(t-1)}(b, B) \end{aligned}$$

for any $z \in [a, b]$. It follows that

$$\begin{aligned} &0 \leq P^{Nt}(b, B) - P^{Nt}(a, B) \\ &\leq \epsilon P^{N(t-1)}(c, B) + (1-\epsilon) P^{N(t-1)}(b, B) - \left[\epsilon \cdot P^{N(t-1)}(c, B) + (1-\epsilon) P^{N(t-1)}(a, B)\right] \\ &= (1-\epsilon) \cdot \left[P^{N(t-1)}(b, B) - P^{N(t-1)}(a, B)\right]. \end{aligned}$$

Continuing on in this manner shows us that

$$0 \leq P^{Nt}(b, B) - P^{Nt}(a, B) \leq (1-\epsilon)^{t-1} \left[P^N(b, B) - P^N(a, B)\right]$$

for any $t \in N_+$. Since $P^N(b, B) - P^N(a, B) \geq 0$ from monotonicity, taking $t \to \infty$ on both sides yields

$$\mu_a^*(B) = \mu_b^*(B).$$

This holds for any $B \in \mathbb{B}'$, so we have

$$\mu_a^*(B) = \mu(B) = \mu_b^*(B)$$

for any $B \in \mathbb{B}'$. μ, μ_a^*, μ_b^* are finite measures that agreee on a π-system generating $\mathcal{Z}$, so that $\mu = \mu^* := \mu_a^* = \mu_b^*$ on $(Z, \mathcal{Z})$. It follows that every weakly convergent subsequence of $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ converges weakly to this distribution μ^*, so that, by the selection theorem for subsequences, $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ itself converges weakly to μ^*. This μ^* is a stationary distribution of P which is concentrated on the stable set $[a, b]$.

Case 2: Arbitrary Initial Distribution

We move onto the case where μ_0 is any initial distribution on $(Z, \mathcal{Z})$. To show our more general result, let μ be a probability measure on $(Z, \mathcal{Z})$ such that $\mu([a, b]) > 0$. Let v be the probability measure on $(Z, \mathcal{Z})$ defined as

$$v(A) = \frac{1}{\mu([a,b])}\mu(A \cap [a,b]) \quad \text{for any } A \in \mathcal{Z}.$$

By design, the mass of v is concentrated on $[a, b]$. Choose any $t \in \mathbb{N}$ and $A \in \mathcal{Z}$. Then,

$$\begin{aligned}(vP^t)(A) = \int_Z P^t(z, A)dv(z) &= \frac{1}{\mu([a,b])} \cdot \int_{[a,b]} P^t(z, A)d\mu(z) \\ &= \frac{1}{\mu([a,b])} \cdot \left(\int_Z P^t(z, A)d\mu(z) - \int_{[a,b]^c} P^t(z, A)d\mu(z)\right) \\ &= \frac{1}{\mu([a,b])} \cdot \left((\mu P^t)(A) - \int_{[a,b]^c} P^t(z, A)d\mu(z)\right)\end{aligned}$$

and we have

$$\begin{aligned}\left|(vP^t)(A) \cdot \mu([a,b]) - (\mu P^t)(A)\right| &= \left|\int_{[a,b]^c} P^t(z, A)d\mu(z)\right| \\ &\leq \int_{[a,b]^c} \left|P^t(z, A)\right|d\mu(z) \leq \mu([a,b]^c).\end{aligned}$$

Now consider the sequence of probability measures $\{\mu_0 P^t\}_{t\in\mathbb{N}}$ on $(Z, \mathcal{Z})$. Fix

some $A \in \mathcal{Z}$ such that $\mu^*(\partial A) = 0$. Choose any $\epsilon \in (0,1)$. By assumption,

$$\lim_{t\to\infty} (\mu_0 P^t)([a,b]) = 1,$$

so there exists some $T \in N_+$ such that

$$(\mu_0 P^t)([a,b]) > 1 - \frac{\epsilon}{3} > 0,$$

or equivalently,

$$(\mu_0 P^t)([a,b]^c) < \frac{\epsilon}{3},$$

for any $t \geq T$. Denote $\mu = \mu_0 P^T$, and define the probability measure v on $(Z, \mathcal{Z})$ as

$$v(B) = \frac{1}{\mu([a,b])}\mu(B \cap [a,b]) \quad \text{for any } B \in \mathcal{Z}.$$

v has mass concentrated on $[a,b]$, and we saw above that

$$\left|(vP^t)(A) \cdot \mu([a,b]) - (\mu P^t)(A)\right| \leq \mu([a,b]^c) = (\mu_0 P^T)([a,b]^c) < \frac{\epsilon}{3}$$

for any $t \in \mathbb{N}$.

Treating v as an initial distribution concentrated on $[a,b]$, by what we showed above vP^t converges weakly to μ^*, so by the Portmanteau theorem we have

$$\lim_{t\to\infty} (vP^t)(A) = \mu^*(A).$$

Therefore, there exists some $T_1 \in N_+$ such that

$$\left|(vP^t)(A) - \mu^*(A)\right| < \frac{\epsilon}{3 \cdot \mu([a,b])}$$

for any $t \geq T_1$. Finally,

$$\mu^*(A) \cdot |\mu([a,b]) - 1| = \mu^*(A) \cdot (\mu_0 P^T)([a,b]^c) < \frac{\epsilon}{3}\mu^*(A) \leq \frac{\epsilon}{3}.$$

Putting these all together,

$$\begin{aligned}
\left|(\mu_0 P^{T+t})(A) - \mu^*(A)\right| &= \left|(\mu P^t)(A) - \mu^*(A)\right| \\
&= \left|\left[(\mu P^t)(A) - \mu([a,b]) \cdot \mu^*(A)\right] + \mu^*(A) \cdot [\mu([a,b]) - 1]\right| \\
&= \left|\left[(\mu P^t)(A) - (vP^t)(A) \cdot \mu([a,b])\right] + \mu([a,b]) \cdot \left[(vP^t)(A) - \mu^*(A)\right] + \mu^*(A) \cdot [\mu([a,b]) - 1]\right| \\
&\leq \left|\mu P^t)(A) - (vP^t)(A)\right| + \mu([a,b]) \cdot \left|(vP^t)(A) - \mu^*(A)\right| + \mu^*(A) \cdot |\mu([a,b]) - 1| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

for any $t \geq T_1$; note that T and T_1 both depend only on ϵ. Defining $T_0 = T + T_1$, we can conclude that

$$\left|(\mu_0 P^t)(A) - \mu^*(A)\right| < \epsilon \quad \text{for any } t \geq T_0$$

for a $T_0 \in N_+$ that depends only on ϵ. This holds for any $\epsilon > 0$, so it follows that

$$\lim_{t \to \infty} (\mu_0 P^t)(A) = \mu^*(A).$$

This in turn holds for any $A \in \mathcal{Z}$ such that $\mu^*(\partial A) = 0$, so by the Portmanteau theorem, $\{\mu_0 P^t\}_{t \in \mathbb{N}}$ converges weakly to μ^*.

Uniqueness of Stationary Distribution

It remains to show that P admits a unique stationary distribution. Suppose that μ is a stationary distribution of P, so that $\mu P^t = \mu$ for any $t \geq \mathbb{N}$. Consider the Markov chain with kernel P started from the initial distribution μ: $\{\mu P^t\}_{t \in \mathbb{N}}$. By the result above, we know that

$$\mu P^t \to \mu^* \quad \text{weaky as } T \to \infty.$$

Defining

$$\mathbb{B}' = \{B \in \mathbb{B} \mid \mu^*(\partial B) = 0\},$$

by the finiteness of μ^* $\mathbb{B}'$ is a π-system generating $\mathcal{Z}$. By the Portmanteau theorem, for any $B \in \mathbb{B}'$ we have

$$\lim_{t\to\infty} \mu P^t(B) = \mu^*(B),$$

and because $\mu P^t = \mu$ on $\mathcal{Z}$, it follows that $\mu(B) = \mu^*(B)$. This shows us that μ and μ^* agree on a π-system generating $\mathcal{Z}$; therefore, $\mu = \mu^*$ on $(Z, \mathcal{Z})$.

Q.E.D.

We say that a Markov kernel P on $(Z, \mathcal{Z})$ satisfies the stable mixing condition if the following hold:

- P satisfies the splitting condition on $[a, b]$ for some $c \in [a, b]$, $\epsilon > 0$ and $N \in N_+$
- $[a, b]$ is a stable set for P and, for any probability measure μ_0 on $(Z, \mathcal{Z})$,

$$\lim_{T\to\infty} (\mu_0 P^T)([a, b]) = 1.$$

The preceding theorem now tells us that a Markov kernel P on a euclidean space that satisfies the Feller property, monotonicity and the stable mixing condition admits a unique stationary distribution to which a Markov chain with kernel P started at any initial distribution converges weakly.

3.6 Parameter-Dependent Markov Chains

In macroeconomic applications, we often work with Markov kernels that are dependent on some parameter, often in the form of prices or macroeconomic aggregates. Naturally, we are interested in how these parameters interact with the Markov kernels of interest. In particular, we want to know when the stationary distribution of the Markov kernel of interest is continuous in the parameter, where continuity is in the sense of the Prohorov metric (the topology of weak convergence).

The basic setting is as follows. Consider a state space $(Z, \mathcal{Z})$ that is a Borel measurable subpsace of some euclidean space $\mathbb{R}^m$, and let Θ be some Borel subspace of $\mathbb{R}^d$ that serves as our parameter space. We let $\{P_\theta\}_{\theta\in\Theta}$ be the parameter-indexed family of Markov kernels on $(Z, \mathcal{Z})$, and $\{\mu_\theta\}_{\theta\in\Theta}$ the corresponding stationary distributions (assuming that they exist). Below we furnish sufficient conditions for

$$\mu_{\theta_n} \to \mu_\theta$$

weakly as $n \to \infty$ given any sequence $\{\theta_n\}_{n\in N_+}$ in Θ that converges to $\theta \in \Theta$. In other words, we furnish sufficient conditions for the mapping $\theta \mapsto \mu_\theta$ to be a continuous function relative to the euclidean topology on $\mathbb{R}^d$ and the topology of weak convergence on $(Z, \mathcal{Z})$ (the topolgy induced by the Prohorov metric on $\mathcal{M}_p(\mathcal{Z})$).

Theorem (Continuity of Stationary Distribution)
Let (Z, d) be a metric space with corresponding Borel σ-algebras $\mathcal{Z}$, and Θ a Borel measurable subset of the euclidean space $\mathbb{R}^d$. Let $\{P_\theta\}_{\theta\in\Theta}$ be a family of parameter-indexed Markov kernel on $(Z, \mathcal{Z})$, and assume that:

i) For any sequence $\{(z_n, \theta_n)\}_{n\in N_+}$ converging to $(z, \theta) \in Z \times \Theta$ in the product metric on $Z \times \Theta$,

$$P_{\theta_n}(z_n, \cdot) \to P_\theta(z, \cdot)$$

weakly as $n \to \infty$.

ii) For any $\theta \in \Theta$, P_θ admits a unique stationary distribution μ_θ.

iii) For any sequence $\{\theta_n\}_{n\in N_+}$ in Θ converging to $\theta_0 \in \Theta$, there exists a compact subset Z_0 of Z such that $\mu_{\theta_n}(Z_0) = 1$ for any $n \in \mathbb{N}$.

Then, for any sequence $\{\theta_n\}_{n\in N_+}$ in the parameter space that converges to θ, μ_{θ_n} converges weakly to μ_θ as $n \to \infty$.

Proof) Let $\{\theta_n\}_{n\in N_+}$ be a sequence in Θ that converges to θ_0. We want to show that $\mu_{\theta_n} \to \mu_{\theta_0}$ weakly.

First, note that, by the third assumption, there exists a compact subset Z_0 such that $\mu_{\theta_n}(Z_0) = 1$ for any $n \in N_+$. This immediately makes $\{\mu_{\theta_n}\}_{n\in N_+}$ a tight sequence of probability measures on $(Z, \mathcal{Z})$. Additionally, because $\{\theta_n\}_{n\in N_+}$ is a convergent sequence, it is bounded. Defining $K = \{\theta_n\}_{n\in N_+} \cup \{\theta_0\}$, K is therefore a bounded set and it is also closed (θ_0 is the unique limit point of $\{\theta_n\}_{n\in N_+}$), so that, by the Heine Borel theorem, K is a compact subset of Θ.

Fix any $f \in C_b(Z, \mathbb{R})$, and define the mapping $g : Z \times \Theta \to \mathbb{R}$ as

$$g(z,\theta) = \int_Z f(z')P_\theta(z, dz')$$

for any $(z,\theta) \in Z \times \Theta$. By assumption, for any $\{(z_n, \hat{\theta}_n)\}_{n\in N_+}$ that converges to some $(z,\theta) \in Z \times \Theta$,

$$P_{\hat{\theta}_n}(z_n, \cdot) \to P_\theta(z, \cdot)$$

weakly as $n \to \infty$, and therefore

$$\lim_{n\to\infty} g(z_n, \hat{\theta}_n) = g(z,\theta).$$

This implies that $g(\cdot)$ is continuous on $Z \times \Theta$, and since $Z_0 \times K$ is a compact subset of $Z \times \Theta$, $g(\cdot)$ is uniformly continuous on $Z_0 \times K$. In other words, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|g(z,\theta) - g(z',\theta')\right| < \epsilon$$

for any $(z,\theta), (z',\theta') \in Z_0 \times K$ such that

$$(d \times \rho)((z,\theta),(z',\theta')) = \max(d(z,z'), \rho(\theta,\theta')) < \delta,$$

where we let ρ be the euclidean metric on $\mathbb{R}^d$. By implication, since

$$(d \times \rho)((z, \theta_n), (z, \theta)) = \rho(\theta_n, \theta_0) \to 0$$

as $n \to \infty$ for any $z \in Z_0$, letting $N \in N_+$ be chosen so that

$$\rho(\theta_n, \theta_0) < \delta \quad \text{for any } n \geq N,$$

we have

$$|g(z, \theta_n) - g(z, \theta_0)| < \epsilon \quad \text{for any } n \geq N,\ z \in Z_0.$$

Taking the supremum over z now shows us that

$$\sup_{z \in Z_0} |g(z, \theta_n) - g(z, \theta_0)| \leq \epsilon \quad \text{for any } n \geq N.$$

Such an $N \in N_+$ exists for any $\epsilon > 0$, so by definition

$$\lim_{n \to \infty} \sup_{z \in Z_0} |g(z, \theta_n) - g(z, \theta_0)| = 0.$$

By the tightness of $\{\mu_{\theta_n}\}_{n \in N_+}$ and Prohorov's theorem, it follows that there exists a subsequence of $\{\mu_{\theta_n}\}_{n \in N_+}$ that converges weakly to some probability measure μ^* on $(Z, \mathcal{Z})$; for notational simplicity, let $\{\mu_{\theta_n}\}_{n \in N_+}$ itself be this subsequence.In addition, because $\mu_{\theta_n}(Z_0) = 1$ for any $n \in N_+$, where Z_0 is compact and thus closed, it follows from the Portmanteau theorem that

$$1 = \limsup_{n \to \infty} \mu_{\theta_n}(Z_0) \leq \mu^*(Z_0) \leq 1,$$

so that $\mu^*(Z_0) = 1$ as well. Now note that

$$\left| \int_Z f d\mu^* - \int_Z f d(\mu^* P_{\theta_0}) \right|$$

$$\leq \left| \int_Z f d\mu^* - \int_Z f d\mu_{\theta_n} \right| + \left| \int_Z f d\mu_{\theta_n} - \int_Z f d(\mu_{\theta_n} P_{\theta_0}) \right| + \left| \int_Z f d(\mu_{\theta_n} P_{\theta_0}) - \int_Z f d(\mu^* P_{\theta_0}) \right|.$$

The first term converges to 0 as $n \to \infty$ by the definition of weak convergence;

as for the third term, because

$$\begin{aligned}
&\left|\int_Z f d(\mu_{\theta_n} P_{\theta_0}) - \int_Z f d(\mu^* P_{\theta_0})\right| \\
&\qquad = \left|\int_Z \int_Z f(z') P_{\theta_0}(z, dz') d\mu_{\theta_n}(z) - \int_Z \int_Z f(z') P_{\theta_0}(z, dz') d\mu^*(z)\right| \\
&\qquad\qquad = \left|\int_Z g(\cdot, \theta_0) d\mu_{\theta_n} - \int_Z g(\cdot, \theta_0) d\mu^*\right|,
\end{aligned}$$

and $g(\cdot, \theta_0)$ is a continuous function on Z, the definition of weak convergence once again tells us that this term converges to 0 as $n \to \infty$. The middle term can be expanded as follows:

$$\begin{aligned}
\left|\int_Z f d\mu_{\theta_n} - \int_Z f d(\mu_{\theta_n} P_{\theta_0})\right| &= \left|\int_Z f d(\mu_{\theta_n} P_{\theta_n}) - \int_Z f d(\mu_{\theta_n} P_{\theta_0}\right| \\
&= \left|\int_Z \int_Z f(z') P_{\theta_n}(z, dz') d\mu_{\theta_n} - \int_Z \int_Z f(z') P_{\theta_0}(z, dz') d\mu_{\theta_n}(z)\right| \\
&= \left|\int_Z g(\cdot, \theta_n) d\mu_{\theta_n} - \int_Z g(\cdot, \theta_0) d\mu^*\right| \\
&\leq \int_Z |g(\cdot, \theta_n) - g(\cdot, \theta_0)| d\mu_{\theta_n} + \left|\int_Z g(\cdot, \theta_0) d\mu_{\theta_n} - \int_Z g(\cdot, \theta_0) d\mu^*\right| \\
&= \int_{Z_0} |g(\cdot, \theta_n) - g(\cdot, \theta_0)| d\mu_{\theta_n} + \left|\int_Z g(\cdot, \theta_0) d\mu_{\theta_n} - \int_Z g(\cdot, \theta_0) d\mu^*\right| \\
&\leq \sup_{z \in Z_0} |g(z, \theta_n) - g(z, \theta_0)| + \left|\int_Z g(\cdot, \theta_0) d\mu_{\theta_n} - \int_Z g(\cdot, \theta_0) d\mu^*\right|,
\end{aligned}$$

where the first equality follows from the fact that μ_{θ_n} is a stationary distribution of P_{θ_n}, and the last equality from the fact that $\mu_{\theta_n}(\cdot)$ is concentrated on Z_0. Since both terms on the right hand side go to 0 as $n \to \infty$ (the first by the uniform continuity result shown above, and the second by the definition of weak convergence), we can see that

$$\left|\int_Z f d\mu_{\theta_n} - \int_Z f d(\mu_{\theta_n} P_{\theta_0})\right| \to 0$$

as $n \to \infty$ as well. Putting everything together, we have just shown that

$$\left| \int_Z f d\mu^* - \int_Z f d(\mu^* P_{\theta_0}) \right| = 0.$$

This holds for any $f \in C_b(Z, \mathbb{R})$, so by the uniqueness of weak limits,

$$\mu^* = \mu^* P_{\theta_0}$$

on $(Z, \mathcal{Z})$, which shows us that μ^* is a stationary distribution of P_{θ_0}. By the third assumption, which tells us that P_{θ_0} admits a unique stationary distribution μ_{θ_0}, it follows that $\mu^* = \mu_{\theta_0}$ on $(Z, \mathcal{Z})$.

We have just shown that, any subsequence of $\{\mu_{\theta_n}\}_{n \in N_+}$ has a further subsequence that converges weakly to the same distribution μ_{θ_0} on $(Z, \mathcal{Z})$. It follows that $\{\mu_{\theta_n}\}_{n \in N_+}$ itself converges weakly to μ_{θ_0}, which completes the proof.

Q.E.D.

3.7 Markov Kernels of Policy Processes

We conclude the chapter on Markov processes by introducing general asymptotic results for policy processes, which will be of primary interest in the sections to come. In recursive macroeconomic models, we focus on two main variables: the state variable and the stochastic shock. Letting the state variable take values in the set A and the shock in the set Z, we assume that the shock follows a Markov process with kernel P, and let the agent choose the optimal level of tomorrow's state variable given the pair of current state and shock (a, z). The mapping $a^* : A \times Z \to A$ that determines tomorrow's state variable given today's state variable and today's stochastic shock is called the policy function.

Owing to the fact that the shock evolves according to a Markov process and the state variable evolves via the deterministic function $a^*(\cdot)$, we can naturally expect the pair of states and shocks to jointly evolve as a Markov process. Indeed, we show below that this is precisely the case, and that the Markov kernel Q corresponding to the state/shock pair inherits many of the most useful properties of the kernel P corresponding to the stochastic shocks. First, we state an important lemma.

Lemma (Generalized Feller Property)
Let (X, d) and (Z, ρ) be separable and complete metric spaces with corresponding Borel σ-algebras $\mathcal{X}$ and $\mathcal{Z}$. Suppose P is a Markov kernel on $(Z, \mathcal{Z})$ with the Feller property. Then, for any $f \in C_b(X \times Z, \mathbb{R})$, the mapping

$$(x, z) \mapsto \int_Z f(x, z')P(z, dz')$$

is also contained in $C_b(X \times Z, \mathbb{R})$.

Proof) Denote $(S, \mathcal{S}) = (X, \mathcal{X}) \bigotimes (Z, \mathcal{Z})$. Choose any $(x, z) \in S$ and let $\{(x_n, z_n)\}_{n \in N_+}$ be a sequence in S converging to (x, z) in the product metric $d \times \rho$. By the definition of the product metric, this also implies that the sequences $\{x_n\}_{n \in N_+}$ and $\{z_n\}_{n \in N_+}$ individually converge to x and z in their respective metrics. Fixing some $f \in C_b(X \times Z, \mathbb{R})$, we want to show that

$$\lim_{n \to \infty} \int_Z f(x_n, z')P(z_n, dz') = \int_Z f(x, z')P(z, dz').$$

We can bound the difference between the two terms above as

$$\begin{aligned}\left|\int_Z f(x_n,z')P(z_n,dz') - \int_Z f(x,z')P(z,dz')\right| &\leq \left|\int_Z f(x_n,z')P(z_n,dz') - \int_Z f(x,z')P(z_n,dz')\right| \\ &\quad + \left|\int_Z f(x,z')P(z_n,dz') - \int_Z f(x,z')P(z,dz')\right| \\ &\leq \int_Z \left|f(x_n,z') - f(x,z')\right| P(z_n,dz') \\ &\quad + \left|\int_Z f(x,z')P(z_n,dz') - \int_Z f(x,z')P(z,dz')\right|.\end{aligned}$$

We first focus on the rightmost term. Since $f(x,\cdot)$ is a bounded and continuous real valued function on Z and $z_n \to z$ as $n \to \infty$ in the metric ρ, by the Feller property we have

$$\lim_{n\to\infty} \int_Z f(x,z')P(z_n,dz') = \int_Z f(x,z')P(z,dz'),$$

so that the rightmost term goes to 0 as $n \to \infty$.

It remains to show that the first term on the right hand side goes to 0 as $n \to \infty$. We proceed in steps.

Step 1: The Set $E_k(\epsilon)$

Denote by μ the probability measure $P(z,\cdot)$, and define $\{\mu_n\}_{n\in N_+} \subset \mathcal{M}_p(\mathcal{Z})$ as $\mu_n = P(z_n,\cdot)$ for any $n \in N_+$. By the Feller property, we know that

$$\mu_n \to \mu \quad \text{weakly as } n \to \infty.$$

For any $n \in N_+$ and $\epsilon > 0$, define

$$A_n(\epsilon) = \{z' \in Z \mid \left|f(x_n,z') - f(x,z')\right| > \epsilon\}.$$

Because $f(x_n,\cdot)$ and $f(x,\cdot)$ are continuous functions on Z, $A_n(\epsilon)$ is an open

subset of Z. Now, for any $k \in N_+$, define

$$E_k(\epsilon) = \bigcup_{n \geq k} A_n(\epsilon).$$

Since the arbitrary union of open sets is open, $E_k(\epsilon)$ is also an open subset of Z. Furthermore, $\bigcap_k E_k(\epsilon) = \emptyset$ by the continuity of f; if $z' \in \bigcap_k E_k(\epsilon)$, then

$$\left| f(x_n, z') - f(x, z') \right| > \epsilon$$

for any $n \in N_+$, so that

$$\liminf_{n \to \infty} \left| f(x_n, z') - f(x, z') \right| \geq \epsilon.$$

But because $f(\cdot, z')$ is continuous and $x_n \to x$ as $n \to \infty$, we must have

$$\limsup_{n \to \infty} \left| f(x_n, z') - f(x, z') \right| = \lim_{n \to \infty} \left| f(x_n, z') - f(x, z') \right| = 0 < \epsilon,$$

a contradiction.

Step 2: The Measure of the Boundary of $E_k(\epsilon)$

The boundary of $E_k(\epsilon)$ is given as

$$\partial E_k(\epsilon) = \overline{E_k(\epsilon)} \cap \overline{E_k(\epsilon)^c} = \partial E_k(\epsilon)^c.$$

The complement of $E_k(\epsilon)$, being the complement of an open set, is closed, and is given as

$$E_k(\epsilon)^c = \bigcap_{n \geq k} A_n(\epsilon)^c = \{z' \in Z \mid \left| f(x_n, z') - f(x, z') \right| \leq \epsilon \quad \forall n \geq k\}.$$

The interior of $E_k(\epsilon)$ then becomes

$$(E_k(\epsilon)^c)^o = \{z' \in Z \mid \left| f(x_n, z') - f(x, z') \right| < \epsilon \quad \forall n \geq k\},$$

and therefore

$$\partial E_k(\epsilon) = \partial E_k(\epsilon)^c = E_k(\epsilon)^c \setminus (E_k(\epsilon)^c)^o$$

$$= \{z' \in Z \mid \Big|f(x_n, z') - f(x, z')\Big| \leq \epsilon \quad \forall n \geq k, \quad \text{with at least one equality}\}.$$

Thus, for fixed $k \in N_+$ and $0 < \epsilon_1 < \epsilon_2$, if

$$z' \in \partial E_k(\epsilon_1)$$

then $z' \notin \partial E_k(\epsilon_2)$, since $z' \in \partial E_k(\epsilon_1)$ requires $|f(x_n, z') - f(x, z')| \leq \epsilon_1 < \epsilon_2$ for any $n \geq k$. In other words,

$$\partial E_k(\epsilon_1) \cap \partial E_k(\epsilon_2) = \emptyset$$

for any distinct $\epsilon_1, \epsilon_2 > 0$. μ is a probability measure and thus can assign positive measure to at most a countable number of sets in the disjoint collection $\{\partial E_k(\epsilon)\}_{\epsilon > 0}$; let $\Delta_k \subset (0, +\infty)$ be the countable collection of $\epsilon > 0$ such that $\mu(E_k(\epsilon)) > 0$.

Step 3: The Main Result

Choose any $\eta > 0$; since $\bigcup_k \Delta_k$ is a countable subset of $(0, +\infty)$ and $(0, \eta)$ is uncountable, there exists some $\epsilon > 0$ such that $0 < \epsilon < \eta$ and $\epsilon \notin \bigcup_k \Delta_k$. Fix any $k \in N_+$. Then, for any $n \geq k$,

$$\begin{aligned}\int_Z \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') &= \int_{A_n(\epsilon)} \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') \\ &\quad + \int_{A_n(\epsilon)^c} \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') \\ &\leq \int_{A_n(\epsilon)} \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') + \epsilon \cdot P(z_n, A_n(\epsilon)^c),\end{aligned}$$

where the inequality follows because $|f(x_n, z') - f(x, z')| \leq \epsilon$ for any $z' \in A_n(\epsilon)^c$. Furthermore, letting $\|f\|_\infty$ be the supremum norm of the continuous function

f, $\|f\|_\infty < +\infty$ because f is bounded, and we can see that

$$\begin{aligned}\int_{A_n(\epsilon)} \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') &\leq 2\|f\|_\infty \cdot P(z_n, A_n(\epsilon) \\ &\leq 2\|f\|_\infty \cdot P(z_n, E_k(\epsilon)),\end{aligned}$$

where the inequality follows from monotonicity because $A_n(\epsilon) \subset E_k(\epsilon)$. As such,

$$\begin{aligned}\int_Z \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') &\leq 2\|f\|_\infty \cdot P(z_n, E_k(\epsilon)) + \epsilon \\ &= 2\|f\|_\infty \cdot \mu_n(E_k(\epsilon)) + \epsilon,\end{aligned}$$

using the definition of μ_n. Since $\mu_n \to \mu$ weakly as $n \to \infty$, and

$$\mu(\partial E_k(\epsilon)) = 0$$

because $\epsilon \notin \Delta_k$, by the Portmanteau theorem we have

$$\lim_{n\to\infty} \mu_n(E_k(\epsilon)) = \mu(E_k(\epsilon)).$$

Thus, taking $n \to \infty$ on both sides of the inequality above, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_Z \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') &\leq 2\|f\|_\infty \Big[\lim_{n\to\infty} \mu_n(E_k(\epsilon))\Big] + \epsilon_k \\ &= 2\|f\|_\infty \mu(E_k(\epsilon)) + \epsilon.\end{aligned}$$

This holds for any $k \in N_+$, and $\{E_k(\epsilon)\}_{k\in N_+}$ is a sequence of sets in $\mathcal{Z}$ that decreases to $\emptyset$, by sequential continuity we have

$$\lim_{k\to\infty} 2\|f\|_\infty \cdot \mu(E_k(\epsilon)) = 0$$

and thus

$$\limsup_{n\to\infty} \int_Z \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') \leq \epsilon < \eta.$$

This in turn holds for any $\eta > 0$, so

$$\lim_{n\to\infty} \int_Z \Big|f(x_n, z') - f(x, z')\Big| P(z_n, dz') = 0.$$

As such,

$$\lim_{n\to\infty}\left[\int_Z \left|f(x_n,z')-f(x,z')\right|P(z_n,dz')+\left|\int_Z f(x,z')P(z_n,dz')-\int_Z f(x,z')P(z,dz')\right|\right]=0$$

and by implication

$$\lim_{n\to\infty}\left|\int_Z f(x_n,z')P(z_n,dz')-\int_Z f(x,z')P(z,dz')\right|=0.$$

This holds for any sequence $\{(x_n,z_n)\}_{n\in N_+}$ converging to (x,z), so the desired result follows.

Q.E.D.

Theorem (Markov Kernel Associated with Policy Processes)
Let Θ be an arbitrary set, and (A,d) and (Z,ρ) separable and complete metric spaces with corresponding σ-algebras $\mathcal{A}$ and $\mathcal{Z}$. Suppose P a Markov kernel on $(Z,\mathcal{Z})$, and let $a^*: A\times Z\times\Theta\to X$ a function such that $a^*(\cdot;\theta)$ is measurable with respect to the product σ-algebra for any $\theta\in\Theta$. For any $\theta\in\Theta$, define the function $Q_\theta:(A\times Z)\times(\mathcal{A}\otimes\mathcal{Z})\to\mathbb{R}$ as

$$Q_\theta((a,z),B)=\int_Z I_B(a^*(a,z;\theta),z')P(z,dz')$$

for any $(a,z)\in A\times Z$ and $B\in\mathcal{A}\otimes\mathcal{Z}$. Then, $\{Q_\theta\}_{\theta\in\Theta}$ is a family of Markov kernels on the product space $(S,\mathcal{S})=(A,\mathcal{A})\otimes(Z,\mathcal{Z})$ such that

$$\int_{A\times Z} f(s')Q_\theta((a,z),ds')=\int_Z f(a^*(a,z;\theta),z')P(z,dz')$$

for any $f\in\mathcal{S}_{+,b}$ and $\theta\in\Theta$.

Proof) We first show that $\{Q_\theta\}_{\theta\in\Theta}$ is a family of Markov kernels on $(S,\mathcal{S})$. Fixing $\theta\in\Theta$, we can see that:

1. For any $B_1 \in \mathcal{A}$ and $B_2 \in \mathcal{Z}$,

$$Q_\theta((a,z), B_1 \times B_2) = P(z, B_2) \cdot I_{B_1} \circ a^*(a,z;\theta)$$

for any $(a,z) \in S$. Since $P(\cdot, B_2) \in \mathcal{Z}_+$ and $a^*(\cdot;\theta)$ is continuous on S, it follows that $Q_\theta(\cdot, B_1 \times B_2) \in \mathcal{S}_+$. Now define the collection $\mathcal{D}$ of $\mathcal{S}$-measurable subsets as

$$\mathcal{D} = \{B \in \mathcal{S} \mid Q_\theta(\cdot, B) \in \mathcal{S}_+\}.$$

We can easily see that it is a λ-system on S:

- $S \in \mathcal{D}$ because $S = A \times Z$ is a measurable rectangle.

- For any $B_1, B_2 \in \mathcal{D}$ such that $B_1 \subset B_2$, $B_2 \setminus B_1 \in \mathcal{S}$ and

$$Q_\theta(\cdot, B_2 \setminus B_1) = Q_\theta(\cdot, B_2) - Q_\theta(\cdot, B_1) \in \mathcal{S}_+,$$

  so that $B_2 \setminus B_1 \in \mathcal{D}$.

- For any $\{B_n\}_{n \in N_+}$ such that $B_n \nearrow B$, $B \in \mathcal{S}$ and

$$Q_\theta(\cdot, B) = \lim_{n \to \infty} Q_\theta(\cdot, B_n) \in \mathcal{S}_+$$

  by the MCT, so that $B \in \mathcal{D}$.

Since $\mathcal{D}$ is a λ-system that contains the π-system of measurable rectangles on S that generates $\mathcal{S}$, it also contains $\mathcal{S}$. Therefore, $Q_\theta(\cdot, B) \in \mathcal{S}_+$ for any $B \in \mathcal{S}$.

2. For any $s \in S$, we can show that $Q_\theta(s, \cdot)$ is a probability measure on $(S, \mathcal{S})$. First, note that $Q_\theta(s, \emptyset)$, being the integral over the zero function, is equal to 0. Now let $\{B_n\}_{n \in N_+}$ be a sequence of disjoint $\mathcal{S}$-measurable sets whose union is $B \in \mathcal{S}$. Then, since

$$I_B(x) = \sum_{n=1}^{\infty} I_{B_n}(x)$$

for any $x \in S$, by the MCT for series we have

$$Q_\theta(s, B) = \sum_{n=1}^{\infty} Q_\theta(s, B_n).$$

Finally, since $I_S(x) = 1$ for any $x \in S$, we have

$$Q_\theta(s, S) = P(z, Z) = 1.$$

Let $(a, z) = s \in S$, and choose a non-negative simple function f on S with canonical form

$$f = \sum_{i=1}^{n} a_i I_{B_i},$$

where $a_1, \cdots, a_n \in \mathbb{R}_+$. By definition, we have

$$\begin{aligned}\int_{A\times Z} f(s')Q_\theta(s, ds') &= \sum_{i=1}^{n} a_i \cdot Q_\theta(s, B_i) = \sum_{i=1}^{n} a_i \cdot \int_Z I_{B_i}(a^*(a, z; \theta), z')P(z, dz') \\ &= \int_Z \left(\sum_{i=1}^{n} a_i I_{B_i}(a^*(a, z; \theta), z') \right) P(z, dz') = \int_Z f(a^*(a, z; \theta), z')P(z, dz').\end{aligned}$$

Now let f be an arbitrary non-negative $\mathcal{S}$-measurable function. Letting $\{f_n\}_{n\in N_+}$ be an increasing sequence of non-negative simple functions with pointwise limit f, by repeated applications of the MCT we have

$$\begin{aligned}\int_{A\times Z} f(s')Q_\theta(s, ds') &= \lim_{n\to\infty} \int_{A\times Z} f_n(s')Q_\theta(s, ds') \\ &= \lim_{n\to\infty} \int_Z f_n(a^*(a, z; \theta), z')P(z, dz') = \int_Z f(a^*(a, z; \theta), z')P(z, dz').\end{aligned}$$

This proves the equivalent representation result.

Q.E.D.

Theorem (Properties of Policy Processes)
Let Θ be a Borel subspace of $\mathbb{R}^d$, and $(A, \mathcal{A})$ and $(Z, \mathcal{Z})$ Borel subspaces of $\mathbb{R}^m$ and $\mathbb{R}^l$.

Suppose P a Markov kernel on $(Z, \mathcal{Z})$, and let $a^* : A \times Z \times \Theta \to X$ a continuous function on $A \times Z \times \Theta$. Let $\{Q_\theta\}_{\theta \in \Theta}$ be the family of Markov kernels defined above. Then, the following hold true:

i) If P possesses the Feller property, then for any sequence $\{(\theta_n, s_n)\}_{n \in N_+}$ in $\Theta \times S$ that converges to some (θ, s),

$$Q_{\theta_n}(s_n, \cdot) \to Q_\theta(s, \cdot)$$

weakly as $n \to \infty$. By implication, each Q_θ possesses the Feller property.

ii) If P is also monotonic and $a^*(\cdot; \theta)$ is increasing on S for any $\theta \in \Theta$, then $\{Q_\theta\}_{\theta \in \Theta}$ is a family of monotonic Markov kernels.

iii) If, in addition to the previous assumptions, $(S, \mathcal{S})$ contains the $m+l$-cell $[c, d]$ and Q_θ satisfies the stable mixing condition on $[c, d]$, then each Q_θ admits a unique stationary distribution μ_θ on $(S, \mathcal{S})$, and

$$\mu_0 Q_\theta^t \to \mu_\theta$$

weakly as $t \to \infty$ for any initial distribution μ_0.

Proof) Suppose P possesses the Feller property. For any $f \in C_b(S, \mathbb{R})$, define the function $g : (S \times \Theta) \times Z \to \mathbb{R}$ as

$$g((s, \theta), z) = \int_Z f(a^*(s; \theta), z') P(z, dz')$$

for any $(s, \theta) \in S \times \Theta$ and $z \in Z$. Since the mapping

$$((s, \theta), z) \mapsto f(a^*(s; \theta), z)$$

is a continuous and bounded function on $(S \times \Theta) \times Z$, and P possesses the Feller property, by the generalized Feller poroperty we can conclude that $g(\cdot)$ is continuous on $(S \times \Theta) \times Z$.

Now choose any sequence $\{(s_n, \theta_n)\}_{n \in N_+}$ on $S \times \Theta$ that converges to (s, θ). We denote $s_n = (a_n, z_n)$ and $s = (a, z)$. As we saw from the previous theorem,

$$\int_S f(s') Q_{\theta_n}(s_n, ds') = \int_Z f(a^*(s_n; \theta_n), z') P(z_n, dz') = g((s_n, \theta_n), z_n)$$

for any $n \in N_+$. The continuity of $g(\cdot)$ now implies that

$$\lim_{n \to \infty} \int_S f(s') Q_{\theta_n}(s_n, ds') = g((s, \theta), z) = \int_Z f(a^*(s; \theta), z') P(z, dz') = \int_S f(s) Q_\theta(s, ds').$$

This holds for any $f \in C_b(S, \mathbb{R})$, so by definition

$$Q_{\theta_n}(s_n, \cdot) \to Q_\theta(s, \cdot)$$

weakly as $n \to \infty$.

Now suppose that P is also monotonic, and that $a^*(\cdot; \theta)$ is an increasing function for any $\theta \in \Theta$. Let $f \in \mathcal{S}_+$ be an increasing function on S, and choose any $s_1, s_2 \in S$ such that $s_1 \leq s_2$. We denote $s_i = (a_i, z_i)$ for $i = 1, 2$.Then, by the preceding theorem,

$$\int_S f(s') Q_\theta(s_1, ds') = \int_Z f(a^*(s_1; \theta), z') P(z_1, dz'),$$

and because P is monotonic and $z_1 \leq z_2$,

$$\int_S f(s') Q_\theta(s_1, ds') \leq \int_Z f(a^*(s_1; \theta), z') P(z_2, dz').$$

$a^*(\cdot; \theta)$ is increasing and so is f, so $f(a^*(s_1; \theta), z') \leq f(a^*(s_2; \theta), z')$ for any $z' \in Z$; the monotonicity of integration tells us that

$$\int_S f(s') Q_\theta(s_1, ds') \leq \int_Z f(a^*(s_2; \theta), z') P(z_2, dz') = \int_S f(s') Q_\theta(s_2, ds'),$$

from which the monotonicity of Q_θ follows.

Suppose that, for each $\theta \in \Theta$, $(S, \mathcal{S})$ contains the $m+l$-cell $[a, b]$ and Q_θ satisfies the stable mixing condition on $[a, b]$. Then, since Q_θ satisfies the Feller property, monotonicity and the stable mixing condition, we can conclude that Q_θ admits a unique stationary distribution μ_θ such that, for any distribution μ_0 on $(S, \mathcal{S})$,

$$\mu_0 Q_\theta^t \to \mu_\theta$$

weakly as $n \to \infty$.

Q.E.D.

Chapter 4

Stochastic Dynamic Problems and Recursive Solutions

So far, we have studied sequential solutions to dynamic optimization problems, which are characterized by Euler equations and transversality conditions. It turns out that, in a problem that is both dynamic and stohastic, deriving sequential solutions is all but impossible. In these cases, we require an alternate approach to solving dynamic optimization problems that involves taking advantage of their recursive structure. Doing so leads to the Bellman equation, which reduces an infinite horizon problem to essentially a two-period problem of allocating resources between today and tomorrow.

In this chapter, we formally set up the sequence problem for an infinite horizon dynamic stochastic optimization problem, define its dual functional problem, and then show how the solutions to each problem relate to one another. Focusing on the functional problem, we then furnish sufficient conditions under which solutions exist and are unique.

4.1 The Sequence Problem

Before delving into stochastic dynamic optimization problems, first recall the setup of the deterministic model. There, the problem was formulated as

$$\max_{\{x_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot U(x_t, x_{t+1})$$

$$\text{subject to} \quad x_0 = \hat{x}_0, \quad x_{t+1} \in \Gamma(x_t) \quad \text{for any } t \geq \mathbb{N},$$

where $\hat{x}_0$ is some initial value, $\beta \in (0,1)$ is a discount factor, U is a bounded function representing instantaneous returns, and $\Gamma(x_t)$ is the set from which the state variable in the next period, x_{t+1}, can be chosen. In this problem, a solution is a sequence $\{x_t\}_{t\in\mathbb{N}}$ that maximizes the objective function subject to the constraints laid out by Γ. We want to consider a stochastic version of this problem, and study the conditions under which it admits a dual representation as a two-period problem.

To this end, we first introduce the stochastic version of the sequence problem, and lay out assumptions under which it is well-posed. Then, we study Markov kernels and chains, which are crucial to the formulation of a dual two-period problem. Afterward, we state the functional equation corresponding to the sequence problem under the appropriate Markov assumptions imposed on the exogenous shock process, and then conclude with the principle of optimality, which reveals how the sequence problem and functional equation are related to one another.

Now consider a stochastic version of the problem. The components of this problem are given as follows:

State Space The state variables takes values in the measurable space $(X, \mathcal{X})$.

Shocks The source of the stochasticity of the problem comes from a sequence $\{z_t\}_{t\in\mathbb{N}}$ of exogenous shocks taking values in some measurable space $(Z, \mathcal{Z})$, and that, for any $t \in N_+$, μ^t is a transition probability kernel from $(Z, \mathcal{Z})$ into $(Z, \mathcal{Z})^t$ such that $\mu^t(z_0, \cdot)$ is the distribution from which $z^t = (z_1, \cdots, z_t)$ is drawn given that the initial shock is z_0.

Let $(S, \mathcal{S})$ be the product of $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$.

Constraints We let the correspondence $\Gamma : S \to X$ represent the instantaneous constraints at time t. That is, given that the state variable is x_t and the exogenous shock is z_t at time t, $\Gamma(x_t, z_t)$ is the constraint set from which the next period's state x_{t+1} can be chosen.

The graph $\mathrm{Gr}(\Gamma)$ of the constraint correspondence $\Gamma : X \times Z \to X$, is defined as

$$\mathrm{Gr}(\Gamma) = \{(x, y, z) \in X \times X \times Z \mid y \in \Gamma(x, z)\}.$$

Assume that $\mathrm{Gr}(\Gamma)$ is Borel measurable with respect to the product Borel σ-algebra on $X \times X \times Z$, and let $\mathcal{G}$ the Borel σ-algebra generated by $\mathrm{Gr}(\Gamma)$.

Returns We assume that the instantaneous return function U depends not only on the value of the current and subsequent period's state variables, but also on the stochastic shock at that period. Formally, the return function is given as $U : \mathrm{Gr}(\Gamma) \to \mathbb{R}$, so that $U(x_t, x_{t+1}, z_t)$ is the return at time t given that the state at time t is x_t, the exogenous shock is z_t, and the state variable chosen for time $t+1$ is $x_{t+1} \in \Gamma(x_t, z_t)$.

As in the deterministic case, we let $\beta \in (0, 1)$ be a discount factor.

A plan is a sequence of measurable functions $\pi = \{\pi_t\}_{t \in \mathbb{N}}$, where π_0 is a constant taking values in X and, for any $t \in N_+$, π_t is a measurable function from Z^t into X. Heuristically, π_0 represents the value of the state variable chosen for time $t = 1$, and $\pi_t(z^t)$ is the value of the state for time $t+1$ that is chosen if the stream of exogenous shocks $z^t = (z_1, \cdots, z_t) \in Z^t$ is realized.

A feasible plan given an initial value $s_0 = (x_0, z_0)$ of the state and the exogenous shock is defined as a plan $\pi = \{\pi_t\}_{t \in \mathbb{N}}$ such that

$$\pi_0 \in \Gamma(s_0)$$

$$\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t) \quad \text{for any } t \in N_+, z^t \in Z^t.$$

In other words, π is feasible if, at any time t, the next period's state variable is chosen in accordance with the constraint. The set of all feasible plans given s_0 is denoted by $\Pi(s_0)$.

At this point, a natural question arises. When are we able to assert that the set of all feasible plans $\Pi(s_0)$ is non-empty for any initial value $s_0 \in S$? It turns out that the following assumption, A1, is sufficient.

A1 The constraint correspondence Γ is non-empty valued, and has a measurable selection, that is, there exists some $h : S \to X$ such that h is measurable relative to $\mathcal{S}$ and $\mathcal{X}$, and $h(x, z) \in \Gamma(x, z)$ for any $(x, z) \in S$.

Under assumptions A1, we can immediately see that there must exist at least one feasible plan under any initial value. To see this, fix $s_0 \in S$ and note that $\pi = \{\pi_t\}_{t\in\mathbb{N}}$ defined as

$$\pi_0 = h(s_0)$$

$$\pi_t(z^t) = h(\pi_{t-1}(z^{t-1}), z_t) \quad \text{for any } t \in N_+, z^t \in Z^t.$$

Since measurability is preserved across compositions, each π_t is a measurable function from Z^t into X. Furthermore, because h is a measurable selection of Γ, $\pi_0 \in \Gamma(s_0)$ and $\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t)$ for any $t \in N_+$ and $z^t \in Z^t$. This shows us that π is a feasible plan starting from s_0 and therefore $\pi \in \Pi(s_0)$.

For future reference, define the set

$$\mathcal{A} = \bigcup_{s_0\in S} (\Pi(s_0) \times \{s_0\})$$

of all pairs of initial values and feasible plans.

Another natural question that arises at this point is whether the objective function is well defined under any initial value and any feasible plan starting from that initial point. In the deterministic case, the objective function given some initial point and a feasible plan was the discounted sum of the instantaneous return functions evaluated under the values of the state variable in that plan. In the stochastic case, given an initial value $s_0 \in S$ and feasible plan $\pi \in \Pi(s_0)$, we want to define the objective function for π in a similar manner, namely, as the limit of the sequence

$$U(x_0, \pi_0, z_0) + \sum_{t=1}^{T} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)$$

as $T \to \infty$. The integrals can be seen as the period-wise expected value of the instantaneous return function given the exogenous shock at time 0, that is, as

$$\mathbb{E}\left[U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \mid z_0\right].$$

We can immediately see that U must be jointly measurable if the integrals are to be well-defined. Furthermore, we want the limit to exist regardless of our choice of U, shock distributions $\mu^t(z_0, \cdot)$ given initial shock z_0, the feasible plan π, and the initial value s_0.

To this end, we impose the following restriction on the instantaneous return function U:

A2 We assume that the return function U is $\mathcal{G}$-measurable. For any initial point $s_0 \in S$, there exists a sequence $\{L_t(s_0)\}_{t\in\mathbb{N}}$ of non-negative reals such that

$$|U(x_0, \pi_0, z_0)| \leq L_0(s_0)$$

$$\left|U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right]\right| \leq L_t(s_0) \quad \text{for any } t \in N_+, z^t \in Z^t$$

for any feasible plan $\{\pi_t\}_{t\in\mathbb{N}} \in \Pi(s_0)$, and

$$\sum_{t=0}^{\infty} \beta^t \cdot L_t(s_0) < +\infty.$$

Under this assumption, the sequence problem is now well-defined.The exact sense in which it is well-defined is given in the theorem below:

Theorem (Well-Defined Sequence Problem)
Let assumptions A1 and A2 hold. Then, for any initial point $s_0 \in S$ and feasible plan $\pi = \{\pi_t\}_{t\in\mathbb{N}} \subset \Pi(s_0)$, the series

$$U(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \mu^t(z_0, dz^t)$$

converges absolutely. The sequence problem is now well-posed as

$$\max_{\{\pi_t\}_{t\in\mathbb{N}}} \quad U(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \mu^t(z_0, dz^t)$$

$$\begin{aligned} \text{subject to} \quad & \pi_0 \in \Gamma(x_0, z_0), \\ & \pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t) \quad \text{for any } t \in N_+, z^t \in Z^t. \end{aligned}$$

Proof) Fix an initial point $s_0 \in S$, and let $\pi = \{\pi_t\}_{t\in\mathbb{N}} \in \Pi(s_0)$ be some feasible plan starting from s_0, whose existence follows from assumption A1.

Under assumption A2, note that

$$\int_{Z^t} \left| U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \right| \mu^t(z_0, dz^t) \leq L_t(s_0) < +\infty,$$

for any $t \in N_+$, so that the mapping

$$z^t \mapsto U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right]$$

is $\mu^t(z_0, \cdot)$-integrable. It now becomes possible to define the non-negative real-valued sequence $\{U_T(s_0)\}_{T \in N_+}$ as

$$U_T(s_0) = |U(x_0, \pi_0, z_0)| + \int_{t=1}^{T} \left| \beta^t \cdot \int_{Z^t} U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \mu^t(z_0, dz^t) \right|$$

for any $T \in N_+$. For any $T \in N_+$, we have

$$\begin{aligned} U_T(s_0) &\leq |U(x_0, \pi_0, z_0)| + \int_{t=1}^{T} \beta^t \cdot \int_{Z^t} \left| U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \right| \mu^t(z_0, dz^t) \\ &\leq \sum_{t=0}^{T} \beta^t \cdot L_t(s_0), \end{aligned}$$

where $\{L_t(s_0)\}_{t \in N_+}$ is a sequence of non-negative reals such that $\sum_{t=0}^{\infty} \beta^t \cdot L_t(s_0) < +\infty$. Taking $T \to \infty$ on both sides yields

$$\lim_{T \to \infty} U_T(s_0) \leq \sum_{t=0}^{\infty} \beta^t \cdot L_t(s_0) < +\infty.$$

This shows us that $\{U_T(s_0)\}_{T \in N_+}$ is a monotonically increasing sequence that is bounded above, and therefore that the series

$$U(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \mu^t(z_0, dz^t)$$

converges absolutely.

Q.E.D.

The value function $V^* : S \to (-\infty, +\infty)$ associated with the sequence problem is defined

as

$$V^*(s_0) = \sup_{\pi \in \Pi(s_0)} F(\pi, s_0)$$

for any $s_0 \in S$, where the funciton $F(\cdot, s_0) : \Pi(s_0) \to \mathbb{R}$ is defined as

$$F(\pi, s_0) = U(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \mu^t(z_0, dz^t)$$

for any $\pi \in \Pi(s_0)$. Note that the value function is always real-valued in this case, since the objective function $F(\cdot, s_0)$ is bounded above by

$$\sum_{t=0}^{\infty} \beta^t \cdot L_t(s_0) < +\infty.$$

A feasible plan $\pi^* \in \Pi(s_0)$ that attains this supremum, that is, satisfies $V^*(s_0) = F(\pi^*, s_0)$, is said to be an optimal plan started from s_0.

It is now clear why a sequential solution to the above problem will be infeasible. Unlike the deterministic case, where we just needed to find a sequence of states that maximized the objective function, here we need to search for a sequence of functions that plans what state variable to choose depending on any realization of the past exogenous shocks. Even if Z were a finite space consisting of two elements, the number of possible paths we must specify in any plan π would increase exponentially fast. In constrast, a recursive approach helps greatly reduce computational burdens because it reduces the problem to a two-period problem and therefore reduces the search for a sequence of increasingly complicated functions to the search for a single function relating today's initial value to tomorrow's realized exogenous shock.

4.2 The Functional Equation

As mentioned earlier, in order to induce a recursive structure for the stochastic problem we require the shocks to also follow a recursive structure, that is, they must follow a Markov process. Therefore, we impose the following assumption on the evolution of the exogenous shocks:

Markov Shocks The exogenous shocks are drawn from some Markov chain with transition probability P. Formally, the sequence of transition probability kernels $\{\mu^t\}_{t\in N_+}$, where each μ^t is a kernel from $(Z,\mathcal{Z})$ into $(Z,\mathcal{Z})^t$, is defined recursively as $\mu^1 = P$ and

$$\mu^t(z_0,A) = \int_{Z^{t-1}} \left(\int_Z I_A(z^{t-1},z_t) P(z_{t-1},dz_t) \right) \mu^{t-1}(z_0,dz^{t-1})$$

for any $(z_0,A) \in Z \times \mathcal{Z}^t$.

Under the above shock specification, we can pose the following functional equation:

$$V(x,z) = \sup_{y\in\Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V(y,z') P(z,dz') \right].$$

This makes it clear why we require the shock process to be modeled as a Markov chain; to reduce an infinite horizon problem to a two-period problem, the shocks from any two adjacent periods must always be related in the same manner, no matter their absolute positions in time.

Given a value function V satisfying the functional equation above, the corresponding policy correspondence $G : S \to X$ is defined as

$$G(x,z) = \left\{ y \in \Gamma(x,z) \mid V(x,z) = U(x,y,z) + \beta \cdot \int_Z V(y,z') P(z,dz') \right\}$$

for any $(x,z) \in S$. Heuristically, $G(x,z)$ collects the values of the state variable next period that allows the supremum to be attained on the right hand side, given that the current state is x and the current shock is z.

If G is non-empty valued and admits a measurable selction, that is, a measurable function $g : X \times Z \to X$ such that $g(x,z) \in G(x,z)$ for any $(x,z) \in S$, then we say that the plan $\pi = \{\pi_t\}_{t\in\mathbb{N}}$ is generated by the policy correspondence G from the initial values $s_0 = (x_0,z_0) \in S$ if there exists a sequence $\{g_t\}_{t\in\mathbb{N}}$ of measurable selections of G such

that

$$\pi_0 = g_0(s_0)$$

$$\pi_t(z^t) = g_t\left[\pi_{t-1}(z^{t-1}), z_t\right] \quad \text{for any } t \in N_+, z^t \in Z^t.$$

It follows that, for any $t \in N_+$, each π_t is a measurable function on Z^t, and by the definition of G,

$$\pi_t(z^t) \in G\left[\pi_{t-1}(z^{t-1}), z_t\right] \subset \Gamma\left(\pi_{t-1}(z^{t-1}), z_t\right)$$

for any $z^t \in Z^t$, so that $\pi \in \Pi(s_0)$, that is, π is a feasible plan starting from s_0.

Heuristically, the policy correspondence G tells us how to choose tomorrow's state variable given the state and shock today in a manner that maximizes the value function today. To see how the functional equation and sequence problem are related to one another, suppose the function V appearing in the functional equation above were the optimal value function of the sequence problem. In this case, choosing tomrrow's state according to G every period would maximize V and therefore solve the sequence problem. Conversely, if the value function of the sequence problem were to solve the functional equation, then the associated optimal policy plan could be broken up into smaller two-period problems, each of which would be chosen according to the policy correpondence.

4.2.1 The Principle of Optimality

The principle of optimality formally relates the functional equation to the sequence problem, and the optimal feasible plan to the policy correspondence. It shows us that the sequence problem can be solved by instead searching for a solution to the functional equation and the associated policy correspondence.

First, we introduce some related concepts and notations. Given any initial point $s_0 = (x_0, z_0) \in S$ and a feasible plan $\pi \in \Pi(s_0)$, the continuation of π from the point $z' \in Z$ is defined as the sequence $C(\pi, z') = \{C_t(\pi, z')\}_{t \in \mathbb{N}}$ of functions such that $C_t(\pi, z') : Z^t \to X$ for any $t \in N_+$ and

$$C_0(\pi, z') = \pi_1(z')$$

$$C_t(\pi, z') = \pi_{t+1}(z', \cdot) \quad \text{for any } t \in N_+.$$

Like its namesake, $C(\pi, z')$ is literally continuing the plan π from the shock $z' \in Z$. Since the sections of measurable functions are measurable, $C(\pi, z')$ is a sequence of measurable functions. Furthermore, by design,

$$C_0(\pi, z') = \pi_1(z') \in \Gamma(\pi_0, z')$$

$$\begin{aligned}[C_t(\pi, z')](z^t) = \pi_{t+1}(z', z^t) &\in \Gamma\left(\pi_t(z', z^{t-1}), z_t\right) \\ &= \Gamma\left([C_{t-1}(\pi, z')](z^{t-1}), z_t\right) \quad \text{for any } t \in N_+, z^t \in Z^t,\end{aligned}$$

so that, by definition, $C(\pi, z') \in \Pi(\pi_0, z')$, that is, it is a feasible plan started from the initial value $(\pi_0, z') \in S$.

We can also think of continuations further into the future. The t-period ahead continuation $C^t(\pi, z^t)$ of π started from $z^t \in Z^t$ is defined as the sequence $\{C^t_s(\pi, z^t)\}_{s \in \mathbb{N}}$ such that

$$C^t_0(\pi, z^t) = \pi_t(z^t)$$

$$C^t_s(\pi, z^t) = \pi_{t+s}(z^t, \cdot) \quad \text{for any } s \in N_+.$$

As above, we can easily show that, for any $z^t \in Z^t$, $C^t(\pi, z^t)$ is a feasible plan started from $(\pi_{t-1}(z^{t-1}), z_t)$.

The following result shows us how the objective function for the sequence problem

can be decomposed in a form similar to the functional equation.

Lemma (Continuation Lemma)
Suppose A1 and A2 hold. Then, for any initial plan $s_0 \in S$ and feasible plan $\pi \in \Pi(s_0)$, we have

$$F(\pi, s_0) = U(x_0, \pi_0, z_0) + \beta \cdot \int_Z F[C(\pi, z'), (\pi_0, z')]P(z_0, dz').$$

Proof) For any $t \geq 2$, we have from the definition of the kernel μ^t the following equation:

$$\int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)$$

$$= \int_Z \int_{Z^{t-1}} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^{t-1}(z_1, d(z_2, \cdots, z_t))P(z_0, dz_1)$$

$$= \int_Z \left[\int_{Z^{t-1}} U\left([C_{t-2}(\pi, z')](z^{t-2}), [C_{t-1}(\pi, z')](z^{t-1}), z_{t-1} \right) \mu^{t-1}(z', dz^{t-1}) \right] P(z_0, dz').$$

For any $T \geq 2$, we therefore have

$$\sum_{t=1}^{T} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)$$

$$= \beta \cdot \int_Z U(\pi_0, \pi_1(z'), z')P(z_0, dz')$$
$$+ \beta \left(\int_Z \left[\sum_{t=2}^{T} \beta^{t-1} \cdot \int_{Z^{t-1}} U\left([C_{t-2}(\pi, z')](z^{t-2}), [C_{t-1}(\pi, z')](z^{t-1}), z_{t-1} \right) \mu^t(z', dz^{t-1}) \right] P(z_0, dz') \right)$$

$$= \beta \cdot \int_Z U(\pi_0, \pi_1(z'), z')P(z_0, dz')$$
$$+ \beta \left(\int_Z \left[\sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right] P(z_0, dz') \right).$$

For any $z' \in Z$, we have

$$\left| \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right|$$

$$\leq \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} \left| U\left[\pi_t(z', z^{t-1}), \pi_{t+1}(z', z^t), z_t\right] \right| \mu^t(z', dz^t).$$

By assumption A2,

$$\left| U\left[\pi_t(z', z^{t-1}), \pi_{t+1}(z', z^t), z_t\right] \right| \leq L_t(s_0)$$

for any $t \in N_+$ and $z^t \in Z^t$, so it follows that

$$\left| \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right|$$

$$\leq \sum_{t=1}^{T-1} \beta^t \cdot L_t(s_0) \leq \underbrace{\sum_{t=1}^{\infty} \beta^t \cdot L_t(s_0)}_{\tilde{L}} < +\infty.$$

It follows from the bounded convergence theorem that

$$\lim_{T\to\infty} \int_Z \left(\sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right) P(z_0, dz')$$

$$= \int_Z \left(\lim_{T\to\infty} \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right) P(z_0, dz')$$

$$= \int_Z \left[F(C(\pi, z'), z') - U(\pi_0, \pi_1(z'), z') \right] P(z_0, dz').$$

Therefore,

$$F(\pi, s_0) - U(x_0, \pi_0, z_0) = \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \mu^t(z_0, dz^t)$$

$$= \beta \cdot \int_Z U(\pi_0, \pi_1(z'), z') P(z_0, dz')$$

$$+ \beta \left(\lim_{T\to\infty} \int_Z \left[\sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U\left([C_{t-1}(\pi, z')](z^{t-1}), [C_t(\pi, z')](z^t), z_t \right) \mu^t(z', dz^t) \right] P(z_0, dz') \right)$$

$$= \beta \cdot \int_Z U(\pi_0, \pi_1(z'), z') P(z_0, dz') + \beta \int_Z \left[F(C(\pi, z'), z') - U(\pi_0, \pi_1(z'), z') \right] P(z_0, dz')$$

$$= \beta \cdot \int_Z F(C(\pi, z'), z') P(z_0, dz').$$

We now have the desired result:

$$F(\pi, s_0) = U(x_0, \pi_0, z_0) + \beta \cdot \int_Z F(C(\pi, z'), z') P(z_0, dz').$$

Q.E.D.

The continuation lemma now allows us to prove the following version of the principle of optimality:

Theorem (Principle of Optimality)
Suppose A1 and A2 hold, and let $V^* : S \to \mathbb{R}$ be the value function associated with the sequence problem. Then, the following hold true:

i) **(Functional Equation Solves Sequence Problem)**
Let $V : S \to [-\infty, +\infty]$ be a solution to the functional equation that satisfies the transversality condition (TVC)

$$\lim_{T \to \infty} \beta^T \cdot \int_{Z^T} V[\pi_{T-1}(z^{T-1}), z_T] \mu^T(z_0, dz^T) = 0$$

for any $s_0 \in S$ and $\pi \in \Pi(s_0)$. Let G be the policy correspondence associated with the solution V. If G is non-empty valued and admits a measurable selection, then $V = V^*$ and any plan π^* generated by G from the initial value s_0 is an optimal plan under s_0.

ii) **(Sequence Problem Solves Functional Equation)**
Suppose that the value function V^* is measurable and solves the functional equation, and let G^* be the policy correspondence associated with V^*. Let $\pi^* \in \Pi(s_0)$ be an optimal plan started from s_0. If G^* is non-empty valued and admits a mea-

surable selection, then

$$\pi_0^* \in G^*(s_0)$$

$$\pi_t^*(z^t) \in G^*(\pi_{t-1}^*(z^{t-1}), z_t) \quad \text{for any } t \in N_+ \text{ and } \mu^t(z_0, \cdot) - \text{a.e. } z^t \in Z^t.$$

Proof) **Functional Equation → Sequence Problem**

First, let V be a solution to the functional problem such that

$$\lim_{T\to\infty} \beta^T \cdot \int_{Z^T} V[\pi_{T-1}(z^{T-1}), z_T]\mu^T(z_0, dz^T) = 0$$

for any $s_0 \in S$ and $\pi \in \Pi(s_0)$. Let G be the policy correspondence associated with V, and suppose that G is non-empty valued and admits a measurable selection.

For any $s_0 \in S$, let $\pi^* \in \Pi(s_0)$ be a plan generated by G from the initial value s_0. By definition, there exists a sequence $\{g_t\}_{t\in\mathbb{N}}$ of measurable functions from S into X such that

$$\pi_0^* = g_0(s_0)$$

$$\pi_t^*(z^t) = g_t(\pi_{t-1}^*(z^{t-1}), z_t) \quad \text{for any } t \in N_+, z^t \in Z^t.$$

By the definition of the policy correspondence, for any $(x, z) \in S$ and $y \in G(x, z)$,

$$V(x, z) = U(x, y, z) + \beta \cdot \int_Z V(y, z')P(z, dz').$$

Since $\pi_0^* = g_0(s_0) \in G(s_0)$,

$$\begin{aligned} V(s_0) &= U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V(\pi_0^*, z')P(z_0, dz') \\ &= U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V(\pi_0^*, z^1)\mu^1(z_0, dz^1). \end{aligned}$$

Suppose, for some $T \geq 1$, that

$$V(s_0) = U(x_0, \pi_0^*, z_0) + \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) \mu^t(z_0, dz^t)$$
$$+ \beta^T \cdot \int_{Z^T} V\left[\pi_{T-1}^*(z^{T-1}), z_T\right] \mu^T(z_0, dz^T).$$

For any $z^T = (z^{T-1}, z_T) \in Z^T$, since $\pi_T^*(z^T) \in G(\pi_{T-1}^*(z^{T-1}), z_T)$, we have

$$V\left[\pi_{T-1}^*(z^{T-1}), z_T\right] = U(\pi_{T-1}^*(z^{T-1}), \pi_T^*(z^T), z_T) + \beta \cdot \int_Z V\left[\pi_T^*(z^T), z'\right] P(z_T, dz'),$$

so that

$$\beta^T \cdot \int_{Z^T} V\left[\pi_{T-1}^*(z^T), z_T\right] \mu^T(z_0, dz^T)$$

$$= \beta^T \cdot \int_{Z^T} U(\pi_{T-1}^*(z^{T-1}), \pi_T^*(z^T), z_T) \mu^T(z_0, dz^T)$$
$$+ \beta^{T+1} \cdot \int_{Z^T} \int_Z V\left[\pi_T^*(z^T), z'\right] P(z_T, dz') \mu^T(z_0, dz^T)$$

$$= \beta^T \cdot \int_{Z^T} U(\pi_{T-1}^*(z^{T-1}), \pi_T^*(z^T), z_T) \mu^T(z_0, dz^T)$$
$$+ \beta^{T+1} \cdot \int_{Z^{T+1}} V\left[\pi_T^*(z^T), z_{T+1}\right] \mu^{T+1}(z_0, dz^{T+1}).$$

Therefore,

$$V(s_0) = U(x_0, \pi_0^*, z_0) + \sum_{t=1}^{T} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) \mu^t(z_0, dz^t)$$
$$+ \beta^{T+1} \cdot \int_{Z^{T+1}} V\left[\pi_T^*(z^T), z_{T+1}\right] \mu^{T+1}(z_0, dz^{T+1}).$$

By induction, it follows that

$$V(s_0) = U(x_0, \pi_0^*, z_0) + \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) \mu^t(z_0, dz^t)$$
$$+ \beta^T \cdot \int_{Z^T} V\left[\pi_{T-1}^*(z^{T-1}), z_T\right] \mu^T(z_0, dz^T).$$

for any $T \in N_+$, and taking $T \to \infty$ on both sides, since

$$\lim_{T\to\infty} \beta^T \cdot \int_{Z^T} V\left[\pi^*_{T-1}(z^{T-1}), z_T\right] \mu^T(z_0, dz^T) = 0,$$

we have

$$V(s_0) = U(x_0, \pi^*_0, z_0) + \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U(\pi^*_{t-1}(z^{t-1}), \pi^*_t(z^t), z_t) \mu^t(z_0, dz^t) = F(\pi^*, s_0).$$

Now choose any $\pi \in \Pi(s_0)$. By definition,

$$V(s_0) = \sup_{y \in \Gamma(s_0)} \left[U(x_0, y, z_0) + \beta \cdot \int_Z V(y, z') P(z_0, dz')\right]$$

$$\geq U(x_0, \pi_0, z_0) + \beta \cdot \int_Z V(\pi_0, z') P(z_0, dz').$$

Suppose, for some $T \geq 1$, that

$$V(s_0) \geq U(x_0, \pi_0, z_0) + \sum_{t=1}^{T-1} \beta^t \cdot \int_{Z^t} U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \mu^t(z_0, dz^t)$$
$$+ \beta^T \cdot \int_{Z^T} V\left[\pi_{T-1}(z^{T-1}), z_T\right] \mu^T(z_0, dz^T).$$

For any $z^T = (z^{T-1}, z_T) \in Z^T$, since $\pi_T(z^T) \in \Gamma(\pi_{T-1}(z^{T-1}), z_T)$ by feasibility, we have

$$V\left[\pi_{T-1}(z^{T-1}), z_T\right] \geq U(\pi_{T-1}(z^{T-1}), \pi_T(z^T), z_T) + \beta \cdot \int_Z V\left[\pi_T(z^T), z'\right] P(z_T, dz'),$$

so that

$$\beta^T \cdot \int_{Z^T} V\left[\pi_{T-1}(z^{T-1}), z_T\right] \mu^T(z_0, dz^T)$$

$$\geq \beta^T \cdot \int_{Z^T} U(\pi_{T-1}(z^{T-1}), \pi_T(z^T), z_T) \mu^T(z_0, dz^T)$$
$$+ \beta^{T+1} \cdot \int_{Z^T} \int_Z V\left[\pi_T(z^T), z'\right] P(z_T, dz') \mu^T(z_0, dz^T)$$

$$= \beta^T \cdot \int_{Z^T} U(\pi_{T-1}(z^{T-1}), \pi_T(z^T), z_T) \mu^T(z_0, dz^T)$$

$$+\beta^{T+1}\cdot\int_{Z^{T+1}}V\left[\pi_T(z^T),z_{T+1}\right]\mu^{T+1}(z_0,dz^{T+1}).$$

Substituting this expression into the inductive hypothesis, we obtain

$$V(s_0)\geq U(x_0,\pi_0,z_0)+\sum_{t=1}^{T}\beta^t\cdot\int_{Z^t}U(\pi_{t-1}(z^{t-1}),\pi_t(z^t),z_t)\mu^t(z_0,dz^t)$$
$$+\beta^{T+1}\cdot\int_{Z^{T+1}}V\left[\pi_T(z^T),z_{T+1}\right]\mu^{T+1}(z_0,dz^{T+1}).$$

By induction, for any $T\in N_+$, we have

$$V(s_0)\geq U(x_0,\pi_0,z_0)+\sum_{t=1}^{T-1}\beta^t\cdot\int_{Z^t}U(\pi_{t-1}(z^{t-1}),\pi_t(z^t),z_t)\mu^t(z_0,dz^t)$$
$$+\beta^{T}\cdot\int_{Z^{T}}V\left[\pi_{T-1}(z^{T-1}),z_{T}\right]\mu^{T}(z_0,dz^{T}).$$

Since the rightmost term goes to 0 as $T\to\infty$ by assumption,

$$V(s_0)\geq U(x_0,\pi_0,z_0)+\sum_{t=1}^{\infty}\beta^t\cdot\int_{Z^t}U(\pi_{t-1}(z^{t-1}),\pi_t(z^t),z_t)\mu^t(z_0,dz^t)=F(\pi,s_0).$$

So far, we have shown that

$$V(s_0)\geq F(\pi,s_0)$$

for any $\pi\in\Pi(s_0)$, and that $V(s_0)=F(\pi^*,s_0)$, where $\pi^*\in\Pi(s_0)$. By definition,

$$V(s_0)=\sup_{\pi\in\Pi(s_0)}F(\pi,s_0)=V^*(s_0),$$

and because $V^*(s_0)$ is attained under the feasible plan π^*, π^* is an optimal plan. Finally, π^* was taken to be any plan generated by G, so that any plan generated by G from s_0 is an optimal plan starting from s_0.

Sequence Problem $\rightarrow$ Functional Equation

Now let the value function V^* of the sequence problem be measurable and solve the functional equation. Denote by G^* the corresponding policy correspondence, which we assume is non-empty valued and admits a measurable selection.

What we want to show is that $\pi_0^* \in G^*(s_0)$ and, for any $t \in N_+$,

$$\pi_t^*(z^t) \in G^*(\pi_{t-1}(z^{t-1}), z_t) \quad \text{for } \mu^t(z_0, \cdot)\text{-almost every } z^t \in Z^t.$$

This is equivalent to showing that

$$V^*(s_0) = U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V^*(\pi_0^*, z') P(z_0, dz')$$

$$V^*[\pi_{t-1}^*(z^{t-1}), z_t] = U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) + \beta \cdot \int_Z V^*[\pi_t^*(z^t), z'] P(z_t, dz')$$
$$\text{for any } t \in N_+ \text{ and } \mu^t(z_0, \cdot) - \text{a.e. } z^t \in Z^t.$$

We first focus on the time 0 equation. By the definition of an optimal plan,

$$V^*(s_0) = F(\pi^*, s_0) = U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z F[C(\pi^*, z'), (\pi_0^*, z')] P(z_0, dz'),$$

where the second equality follows from the preceding lemma. By definition,

$$V^*(\pi_0^*, z') \geq F[C(\pi^*, z'), (\pi_0^*, z')]$$

for any $z' \in Z$, so we have

$$V^*(s_0) \leq U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V^*(\pi_0^*, z') P(z_0, dz').$$

Since G^* is the policy correspondence for the solution V^* to the functional equation, by the proof that the functional equation solves the sequence problem, any plan generated by G^* from the initial value s_0 is optimal under s_0. Let

$g : X \times Z \to X$ be a measurable selection of G^*, and define $\pi^g \in \Pi(s_0)$ as

$$\pi_0^g = \pi_0^*$$

$$\pi_t^g(z^t) = g[\pi_{t-1}^g(z^{t-1}), z_t] \quad \text{for any } t \in N_+, z^t \in Z^t,$$

For any $z' \in Z$, note that the continuation $C(\pi^g, z')$ satisfies

$$C_0(\pi^g, z') = \pi_1^g(z') = g(\pi_0^*, z')$$

$$[C_t(\pi^g, z')](z^t) = \pi_{t+1}^g(z', z^t) = g(\pi_t^g(z', z^{t-1}), z_t) = g\left([C_{t-1}(\pi^g, z')](z^{t-1}), z_t\right)$$
$$\text{for any } t \in N_+, z^t \in Z^t,$$

so $C(\pi^g, z')$ is a plan generated by G^* from the initial value $(\pi_0^*, z') \in S$; therefore, it is optimal under (π_0^*, z'), and as such

$$V^*(\pi_0^*, z') = F[C(\pi^g, z'), (\pi_0^*, z')]$$

for any $z' \in Z$. Finally, using the fact that $\pi^g \in \Pi(s_0)$, we can see that

$$V^*(s_0) \geq F(\pi^g, s_0) = U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z F[C(\pi^g, z'), (\pi_0^*, z')]P(z_0, dz')$$

$$= U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V^*(\pi_0^*, z')P(z_0, dz').$$

Putting the two inequalities together now implies that

$$V^*(s_0) = U(x_0, \pi_0^*, z_0) + \beta \cdot \int_Z V^*(\pi_0^*, z')P(z_0, dz').$$

Note that this also implies that

$$\int_Z V^*(\pi_0^*, z')P(z_0, dz') = \int_Z F[C(\pi^*, z'), (\pi_0^*, z')]P(z_0, dz'),$$

or equivalently,

$$\int_Z \left(V^*(\pi_0^*, z') - F[C(\pi^*, z'), (\pi_0^*, z')]\right) P(z_0, dz') = 0.$$

Since the integrand is non-negative, by the vanishing property for non-negative

functions

$$V^*(\pi_0^*, z') = F[C(\pi^*, z'), (\pi_0^*, z')]$$

for $P(z_0, \cdot)$-a.e. $z' \in Z$.

Now suppose that, for some $t \geq 1$,

$$V^*(\pi_{t-1}^*(z^{t-1}), z_t) = F[C^t(\pi^*, z^t), (\pi_{t-1}^*(z^{t-1}), z_t)]$$

for $\mu^t(z_0, \cdot)$-a.e. $z^t \in Z^t$; let the set on which the equality holds be denoted $A_t(z_0)$. Fix some $z^t \in A^t(z_0)$. Then, by the preceding lemma,

$$V^*(\pi_{t-1}^*(z^{t-1}), z_t) = U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) + \beta \cdot \int_Z F[C^{t+1}(\pi^*, z^{t+1}), (\pi_t^*(z^t), z_{t+1}) P(z_t, dz_{t+1})$$

$$\leq U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) + \beta \cdot \int_Z V^*(\pi_t^*(z^t), z_{t+1}) P(z_t, dz_{t+1}).$$

Now define $\pi^g \in \Pi(\pi_{t-1}^*(z^{t-1}), z_t)$ as

$$\pi_0^g = \pi_t^*(z^t)$$

$$\pi_s^g(z_{t+1}, \cdots, z_{t+s}) = g\left(\pi_{t-1}^g(z_{t+1}, \cdots, z_{t+s-1}), z_{t+s}\right) \quad \text{for any } s \in N_+, (z_{t+1}, \cdots, z_{t+s}) \in Z^s.$$

Then, $\pi^g \in \Pi(\pi_{t-1}^*(z^{t-1}), z_t)$, and the continuation $C(\pi^g, z_{t+1})$ is, as before, a plan generated by G^* from the initial value $(\pi_t^*(z^t), z_{t+1})$. As such, by the result that the functional equation solves the sequence problem, $C(\pi^g, z_{t+1})$ is an optimal plan started from $(\pi_t^*(z^t), z_{t+1})$, which implies that

$$V^*(\pi_t^*(z^t), z_{t+1}) = F[C(\pi^g, z_{t+1}), (\pi_t^*(z^t), z_{t+1})],$$

so that

$$V^*(\pi_{t-1}^*(z^{t-1}), z_t) \geq F[\pi^g, (\pi_{t-1}^*(z^{t-1}), z_t)]$$

$$= U(\pi_{t-1}^*(z^{t-1}), \pi_t^*(z^t), z_t) + \beta \cdot \int_Z F[C(\pi^g, z_{t+1}), (\pi_t^*(z^t), z_{t+1})] P(z_t, dz_{t+1})$$

$$= U(\pi^*_{t-1}(z^{t-1}), \pi^*_t(z^t), z_t) + \beta \cdot \int_Z V^*(\pi^*_t(z^t), z_{t+1}) P(z_t, dz_{t+1}).$$

Putting together the two inequalities yields

$$V^*(\pi^*_{t-1}(z^{t-1}), z_t) = U(\pi^*_{t-1}(z^{t-1}), \pi^*_t(z^t), z_t) + \beta \cdot \int_Z V^*(\pi^*_t(z^t), z_{t+1}) P(z_t, dz_{t+1}),$$

and also that

$$\int_Z \left(V^*(\pi^*_t(z^t), z_{t+1}) - F[C^{t+1}(\pi^*, z^{t+1}), (\pi^*_t(z^t), z_{t+1})] \right) P(z_t, dz_{t+1}) = 0.$$

Integrating both sides with respect to $\mu^t(z_0, \cdot)$ over z^t reveals that

$$\begin{aligned} 0 &= \int_{A^t(z_0)} \int_Z \left(V^*(\pi^*_t(z^t), z_{t+1}) - F[C^{t+1}(\pi^*, z^{t+1}), (\pi^*_t(z^t), z_{t+1})] \right) P(z_t, dz_{t+1}) \mu^t(z_0, dz^t) \\ &= \int_{Z^t} \int_Z \left(V^*(\pi^*_t(z^t), z_{t+1}) - F[C^{t+1}(\pi^*, z^{t+1}), (\pi^*_t(z^t), z_{t+1})] \right) P(z_t, dz_{t+1}) \mu^t(z_0, dz^t) \\ &= \int_{Z^{t+1}} \left(V^*(\pi^*_t(z^t), z_{t+1}) - F[C^{t+1}(\pi^*, z^{t+1}), (\pi^*_t(z^t), z_{t+1})] \right) \mu^{t+1}(z_0, dz^{t+1}), \end{aligned}$$

where the second equality follows because $\mu^t(z_0, A_t(z_0)^c) = 0$. Therefore, by the vanishing property for non-negative functions,

$$V^*(\pi^*_t(z^t), z_{t+1}) = F[C^{t+1}(\pi^*, z^{t+1}), (\pi^*_t(z^t), z_{t+1})]$$

for $\mu^{t+1}(z_0, \cdot)$-a.e. $z^{t+1} \in Z^{t+1}$.

The desired result now follows by induction.

Q.E.D.

The principle of optimality is most useful in the following situation. Suppose that there exists a solution V to the functional problem with policy correspondence G, which is non-empty valued and admits a measurable selection. Then, we can see that V is exactly the value function V^* of the sequence problem, and that any plan generated by G is optimal. This dovetails nicely into the second part of the principle of optimality. In this case, since $V^* = V$, V^* is measurable, solves the functional equation, and the

policy correspondence associated with V^* is non-empty valued and admits a measurable selection. The second part of the theorem then tells us that any optimal plan is almost surely generated by the policy correspondence G.

In other words, if we can only show that there exists a solution V to the functional equation, and that the associated policy correspondence is non-empty valued and admits a measurable selection, then any plan generated by G is not only optimal, but also any optimal plan must behave as if it were generated by G. It is this duality between the sequence problem and the functional equation that allows us to focus exclusively on the functional equation going forward. The second part of the theorem cannot be understated; without it, there could exist multiple optimal plans that have nothing to do with the policy correspondence.

4.3 The Case of Bounded Returns

In the previous section, we established the duality of the sequence problem and the functional equation. Specifically, we saw that, under our maintained assumptions, a solution to the functional equation with a non-empty policy correspondence that has a measurable selection is the value function for the sequence problem, and that, in this case, any optimal plan must be almost surely generated by the policy correpsondence in question. This provides a rationale for focusing on solutions to the functional equation.

The first question we must answer is under which conditions a solution to the functional equation exists. Furthermore, we also want to establish that it is unique; if we can do so, then this unique solution will be the unique solution to the sequence problem in light of the principle of optimality. After establishing existence and uniqueness, we want to search for sufficient conditions under which the solution we obtained possesses useful properties such as continuity, monotonicity and differentiability.

We initially focus on the case where the instantaneous return function U is bounded. This can be achieved in most cases by assuming continuity and restricting our attention to a compact state space X. Boundedness is an especially attractive assumption because it immediately implies that assumption A2 holds:

Lemma Suppose that the instantaneous return function U is bounded. In this case, if assumption A1 is satisfied, then so is assumption A2.

Proof) Suppose that there exists some $M < +\infty$ such that $|U(x, x', z)| \leq M$ for any $(x, x', z) \in \mathrm{Gr}(\Gamma)$, and let assumption A1 hold. Then, for any initial value $s_0 \in S$, the set $\Pi(s_0)$ of feasible plans started from s_0 is non-empty. Furthermore, we can define the sequence $\{L_t(s_0)\}_{t\in\mathbb{N}}$ as

$$L_t(s_0) = M \quad \text{for any } t \in \mathbb{N}.$$

It follows that, for any $\pi = \{\pi_t\}_{t\in\mathbb{N}} \in \Pi(s_0)$,

$$|U(x_0, \pi_0, z_0)| \leq M = L_0(s_0)$$

$$\left|U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right]\right| \leq M = L_t(s_0) \quad \text{for any } t \in N_+, z^t \in Z^t,$$

and

$$\sum_{t=0}^{\infty} \beta^t \cdot L_t(s_0) = M \cdot \sum_{t=0}^{\infty} \beta^t = \frac{M}{1-\beta} < +\infty.$$

This shows us that assumption A2 is satisfied.

Q.E.D.

Therefore, under assumption A1 and the measurability and boundedness of U, the principle of optimality holds by the preceding theorem. Now we can furnish additional assumptions under which a continuous solution to the functional equation exists and is unique.

4.3.1 Existence and Uniqueness

To establish the existence of a solution to the functional equation, we make use of the fact that the operation

$$f \mapsto \left((x,z) \mapsto \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right] \right)$$

for any real valued function f is an operation on the space of real-valued functions. If this operation, under the appropriate norm, is established as a contraction, then Banach's fixed point theorem implies that there exists a unique fixed point for this operation, that is, a function V such that

$$V(x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right]$$

for any $(x,z) \in S$. This V will be the unique continuous solution to the functional equation that we desire.

To this end, we work with the following assumptions:

B1 (X,d) and (Z,ρ) be separable and complete metric spaces, with corresponding Borel σ-algebras $\mathcal{X}$ and $\mathcal{Z}$.

B2 The Markov kernel P has the Feller property.

B3 The constraint correspondence $\Gamma : X \times Z \to X$ is non-empty, compact valued and continuous. The continuity and compact-valuedness of Γ implies that $\mathrm{Gr}(\Gamma)$ is a closed set by the closed graph theorem.

B4 The instantaneous return function $U : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ is bounded and continuous with respect to the product metric on $X \times X \times Z$ induced by d and ρ.

Note that assumptions B1 to B4 imply assumptions A1 and A2. The completeness and separability of the metric space (X,d), alongside the fact that Γ is non-empty compact valued and continuous, implies via the Kuratowski-Ryll-Nardzewski measurable selection theorem that Γ admits a measurable selection, so that assumption A1 holds. Meanwhile, assumption B4 implies assumption A2 by the preceding lemma, and because continuous functions are measurable.

The assumption that is completely new to this section is assumption B2, which concerns the continuity property of the Markov kernel P. This condition is satisfied via the dominated convergence theorem if there exists a σ-finite measure v on $(Z,\mathcal{Z})$ such that $P(z,\cdot) << v$ for any $z \in Z$, so that P admits a family of densities $\{p(\cdot \mid z)\}_{z \in Z}$ with respect to v, and moreover, if this family is dominated by a v-integrable function.

The power of assumption B2 can be seen once we define the functional equation operator. Let $C_b(S,\mathbb{R})$ be the set of all bounded and continuous real functions on S, for any $f \in C_b(S,\mathbb{R})$ we define the function $T(f) : S \to (-\infty,+\infty]$ as

$$[T(f)](x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right]$$

for any $(x,z) \in S$. Note that the integral on the right hand side is well-defined for any $y \in \Gamma(x,z)$ since f is bounded and continuous (therefore measurable). We want to show

that the function $T(f)$ is real-valued, bounded and continuous on S. To do so, we want to use Berge's maximum theorem, which applies for constraint sets that are non-empty compact valued and continuous, and objective functions that are continuous. The former is guaranteed by assumption B3, and assumption B2, along with assumption B4, ensures that the latter holds. Specifically, under assumption B2, we can establish that

$$(y, z) \mapsto \int_Z f(y, z')P(z, dz')$$

is a continuous mapping on S. This result is stated as a lemma below:

Lemma Under assumptions B1 and B2, we can define $M : C_b(S, \mathbb{R}) \to C_b(S, \mathbb{R})$ as

$$[M(f)](x, z) = \int_Z f(x, z')P(z, dz')$$

for any $(x, z) \in S$ and $f \in C_b(S, \mathbb{R})$.

Proof) For any $f \in C_b(S, \mathbb{R})$, the boundedness and measurability of f tells us that the function $M(f) : S \to \mathbb{R}$ is well-defined as

$$[M(f)](x, z) = \int_Z f(x, z')P(z, dz')$$

for any $(x, z) \in S$. The main result we must show is that $M(f)$ is itself continuous and bounded. Boundedness is easy to see; if $|f| \leq m$ on S for some $m > 0$,

$$|[M(f)](x, z)| \leq \int_Z \left|f(x, z')\right| P(z, dz') \leq m$$

for any $(x, z) \in S$, so that $M(f)$ is bounded by m as well.

The continuity part now follows from the generalized Feller property, since P possesses the Feller property.

Q.E.D.

The above lemma, in conjunction with Berge's maximum theorem, immediately implies that the operator T is continuous. This is the content of the next result:

Lemma (Continuity of Functional Equation)
Under assumptions B1 to B4, the operator T defined as

$$[T(f)](x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right]$$

for any $f \in C_b(S,\mathbb{R})$ and $(x,z) \in S$ is a mapping from $C_b(S,\mathbb{R})$ into $C_b(S,\mathbb{R})$.

Proof) Fix any $f \in C_b(S,\mathbb{R})$, and define the function $F : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ as

$$F(x,y,z) = U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') = U(x,y,z) + \beta \cdot [M(f)](y,z)$$

for any $(x,y,z) \in \mathrm{Gr}(\Gamma)$. Since $M(f)$ is a continuous function on S by the preceding lemma and U is continuous by assumption B4, F is continuous on $\mathrm{Gr}(\Gamma)$. Furthermore, Γ is a non-empty compact valued and continuous correspondence. $T(f) : S \to (-\infty, +\infty]$ as defined above is then the value function corresponding to the maximization problem

$$\max_{y \in \Gamma(x,z)} \quad F(x,y,z)$$

for any $(x,z) \in S$. Therefore, by Berge's maximum theorem, $T(f)$ is real valued and continuous on S.

Q.E.D.

So far, we have established that the operator T defined above maps $C_b(S,\mathbb{R})$ into $C_b(S,\mathbb{R})$. Under the supremum norm $\|\cdot\|_\infty$ and the metric d_∞ on $C_b(S,\mathbb{R})$ induced by the supremum norm, the space $C_b(S,\mathbb{R})$ becomes a complete metric space. Thus, to apply Banach's fixed point theorem, we need only show that T is a contraction mapping on $C_b(S,\mathbb{R})$. To this end, we can use Blackwell's sufficient conditions:

Lemma (Functional Equation Operator is a Contraction Mapping) Under assumptions B1 to B4, the operator $T : C_b(S, \mathbb{R}) \to C_b(S, \mathbb{R})$ defined as

$$[T(f)](x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right]$$

for any $f \in C_b(S, \mathbb{R})$ and $(x,z) \in S$ is a contraction mapping given the supremum norm $\|\cdot\|_\infty$.

Proof) Recall that the Blackwell sufficient conditions are:

i) **(Monotonicity)** For any $f, g \in C_b(S, \mathbb{R})$ such that $f \leq g$, $T(f) \leq T(g)$.

ii) **(Discounting)** There exists a $b \in (0,1)$ such that, for any $f \in C_b(S, \mathbb{R})$ and $c \geq 0$,

$$T(f + c) \leq T(f) + b \cdot c.$$

We first verify that T satisfies the monotonicity property. For any $f, g \in C_b(S, \mathbb{R})$ such that $f \leq g$, by the monotonicity of integration

$$[M(f)](y,z) = \int_Z f(y,z') P(z,dz') \leq \int_Z g(y,z') P(z,dz') = [M(g)](y,z)$$

for any $(y,z) \in X \times Z$. Therefore,

$$U(x,y,z) + \beta \cdot [M(f)](y,z) \leq U(x,y,z) + \beta \cdot [M(g)](y,z)$$

for any $(x,y,z) \in \mathrm{Gr}(\Gamma)$, so that

$$\begin{aligned} [T(f)](x,z) &= \sup_{y \in \Gamma(x,z)} [U(x,y,z) + \beta \cdot [M(f)](y,z)] \\ &\leq \sup_{y \in \Gamma(x,z)} [U(x,y,z) + \beta \cdot [M(g)](y,z)] = [T(g)](x,z) \end{aligned}$$

for any $(x,z) \in S$.

Moving onto the discounting property, choose any $f \in C_b(S,\mathbb{R})$ and $c \geq 0$. Then,

$$[M(f+c)](y,z) = \int_Z \Big(f(y,z')+c\Big)\, P(z,dz') = \int_Z f(y,z')P(z,dz') + c = [M(f)](y,z) + c$$

for any $(y,z) \in X \times Z$, so that

$$U(x,y,z) + \beta \cdot [M(f+c)](y,z) = U(x,y,z) + \beta \cdot [M(f)](y,z) + \beta \cdot c$$

for any $(x,y,z) \in \mathrm{Gr}(\Gamma)$. It follows that

$$\begin{aligned}[T(f+c)](x,z) &= \sup_{y \in \Gamma(x,z)} [U(x,y,z) + \beta \cdot [M(f)](y,z) + \beta \cdot c] \\ &= \sup_{y \in \Gamma(x,z)} [U(x,y,z) + \beta \cdot [M(f)](y,z)] + \beta \cdot c = [T(f)](x,z) + \beta \cdot c\end{aligned}$$

for any $(x,z) \in S$. Thus, discounting holds with discount rate $b = \beta$. This proves, in light of the Blackwell sufficient conditions, that T is a contraction mapping on $C_b(S,\mathbb{R})$.

Q.E.D.

We are now ready to state the main existence and uniqueness result. So far, we have shown that the operator T associated with the functional equation is a contraction mapping on the complete metric space $(C_b(S,\mathbb{R}), d_\infty)$. Therefore, we can apply Banach's fixed point theorem to establish the existence of a unique fixed point for T; in fact, it also gives us a simple algorithm to find this fixed point. Finally, another application of Berge's maximum theorem shows us that the associated policy correspondence function is non-empty, compact valued and upper hemicontinuous, so that it admits a measurable selection in light of the measurable selection theorem. We formally state and prove these results below:

Theorem (Existence and Uniqueness of Solution to Functional Equation) Under assumptions B1 to B4, the following hold true:

i) **(Existence and Uniqueness)**
There exists a unique solution on $C_b(S,\mathbb{R})$ to the functional equation, that is, a unique function $V^* \in C_b(S,\mathbb{R})$ such that

$$V^*(x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V^*(y,z') P(z,dz') \right]$$

for any $(x,z) \in S$.

ii) **(Iterative Scheme for finding V^*)**
For any initial value function $V^{(0)} \in C_b(S,\mathbb{R})$ and the sequence $\{V^{(i)}\}_{i \in \mathbb{N}}$ defined as $V^{(i+1)} = T(V^{(i)})$ for any $i \in \mathbb{N}$,

$$\lim_{i \to \infty} \left\| V^{(i)} - V^* \right\|_\infty = 0.$$

iii) **(V^* satisfies Transversality Condition)**
For any initial point $s_0 \in S$ and feasible plan $\pi \in \Pi(s_0)$, we have

$$\lim_{t \to \infty} \beta^t \int_{Z^t} V^*[\pi_{t-1}(z^{t-1}), z_t] \mu^t(z_0, dz^t) = 0.$$

iv) **(Properties of Policy Correspondence)**
The associated policy correspondence $G^* : X \times Z \to X$ defined as

$$G^*(x,z) = \{ y \in \Gamma(x,z) \mid V^*(x,z) = U(x,y,z) + \beta \cdot \int_Z V^*(y,z') P(z,dz') \}$$

for any $(x,z) \in S$ is non-empty compact valued and upper hemicontinuous. Furthermore, G^* admits a measurable selection.

Proof) Letting the operator T be defined as above, we proved that T is a contraction mapping on the complete metric space $(C_b(S,\mathbb{R}), d_\infty)$. By Banach's fixed point

theorem, there exists a unique $V^* \in C_b(S, \mathbb{R})$ such that

$$T(V^*) = V^*,$$

or more precisely,

$$V^*(x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V^*(y,z') P(z,dz') \right]$$

for any $(x,z) \in S$. Furthermore, the theorem tells us that we can find V^* iteratively by starting at some initial function $V^{(0)} \in C_b(S, \mathbb{R})$ and defining the sequence $\{V^{(i)}\}_{i \in \mathbb{N}}$ as $V^{(i+1)} = T(V^{(i)})$ for any $i \in \mathbb{N}$. Then, V^* is the limit of the sequence $\{V^{(i)}\}_{i \in \mathbb{N}}$ in the supremum norm $\|\cdot\|_\infty$.

The claim about the transversality condition for V^* follows almost immediately from the boundedness of V^*. Given any initial point $s_0 \in S$ and feasible plan $\pi \in \Pi(s_0)$, note that, for any $t \in N_+$,

$$\begin{aligned} \left| \beta^t \int_{Z^t} V^*[\pi_{t-1}(z^{t-1}), z_t] \mu^t(z_0, dz^t) \right| &\leq \beta^t \cdot \int_{Z^t} \left| V^*[\pi_{t-1}(z^{t-1}), z_t] \right| \mu^t(z_0, dz^t) \\ &\leq \beta^t \|V^*\|_\infty \cdot \left(\int_{Z^t} 1 \mu^t(z_0, dz^t) \right) = \beta^t \|V^*\|_\infty. \end{aligned}$$

Since $\|V^*\|_\infty < +\infty$ by the boundedness of V^*, and $\beta \in (0,1)$, the rightmost term goes to 0 as $t \to \infty$, showing us that

$$\lim_{t \to \infty} \beta^t \int_{Z^t} V^*[\pi_{t-1}(z^{t-1}), z_t] \mu^t(z_0, dz^t) = 0.$$

Note that V^* is the value function corresponding to the maximization problem

$$\max_{y \in \Gamma(x,z)} \quad [U(x,y,z) + \beta \cdot [M(V^*)](y,z)],$$

where the objective function $U(x,y,z) + \beta \cdot [M(V^*)](y,z)$ is continuous on $\mathrm{Gr}(\Gamma)$ and the constraint correspondence Γ is non-empty compact valued and continuous on $X \times Z$. Therefore, by Berge's maximum theorem, the set of maximizers

$G^* : S \to X$ defined as

$$G^*(x,z) = \{y \in \Gamma(x,z) \mid V^*(x,z) = U(x,y,z) + \beta \cdot [M(V^*)](y,z)\}$$

for any $(x,z) \in S$ is non-empty compact valued and upper hemicontinuous. Since (X,d) is a complete and separable metric space, by the Kuratowski-Ryll-Nardzewski measurable selection theorem, G^* admits a measurable selection.

Q.E.D.

It is important to keep in mind what this theorem does and does not say. It does say that there exists a unique continuous and bounded function that solves the functional equation, and that the corresponding policy correspondence satisfies the conditions needed for the principle of optimality to hold (non-empty valuedness and existence of a measurable selection). Therefore, we can apply the principle of optimality to conclude that the value function of the sequence problem solves the functional equation, that any plan generated by the policy correspondence is an optimal plan (and thus that the set of optimal plans is non-empty), and finally that any optimal plan is almost surely generated by this policy correspondence.

On the other hand, it does not say that the functional equation has a unique limit period. Since we are only looking at the class of continuous bounded solutions to the functional equations, it is possible that there exist other, non-continuous or non-bounded solutions to the functional equation. Furthermore, because the policy correspondence remains a correspondence, not a function, we are unable to assert the uniqueness of an optimal plan as well. This is why we now study conditions under which the policy correspondence becomes a policy function, and more generally, under which we can apply the usual tools of optimization to study optimal plans.

4.3.2 Monotonicity and Concavity of the Value Function

In the previous section, we showed that under assumptions B1 to B4, there exists a unique continuous and bounded solution V^* to the functional equation, and that the associated policy correspondence G^* is non-empty compact valued and upper hemicontinuous. Here we furnish additional assumptions under which V^* and G^* possess other useful properties, such as monotonicity in the state variable in the case of V^* and single-valuedness in the case of G^* (which automatically establishes the continutiy of the policy function).

The workhorse theorem used to derive all these properties about V^* is the following result on contraction mappings. Roughly speaking, it shows us that, if a subset of $C_b(S, \mathbb{R})$ possessing some desirable property "contracts" under the functional equation operator T defined in the previous section, then the unique fixed point V^* also satisfies this property.

Lemma (Properties Theorem)
Let (X, d) be a complete metric space, and $\phi : X \to X$ a contraction mapping on X. Letting the unique fixed point of ϕ be denoted by $v^* \in X$, if $\phi(A) \subset A$ for some non-empty closed subset A of X, then $v^* \in A$. Similarly, if $\phi(A) \subset A' \subset A$ for some non-empty closed subset A of X and an arbitrary non-empty subset A' of X, then $v^* \in A'$.

Proof) Let A be a non-empty closed subset of X such that $\phi(A) \subset A$. Choosing some $v_0 \in A$, Banach's fixed point theorem tells us that the sequence $\{\phi^n(v_0)\}_{n \in N_+}$ converges in d to the fixed point v^*. Since $\phi(A) \subset A$ and the initial point $v_0 \in A$, we can see that this entire sequence is contained in A, and since A is a closed set, the limit v^* of this sequence is also contained in A, which establishes $v^* \in A$.

On the other hand, suppose A is a non-empty closed subset of X, and A' an arbitrary non-empty subset of X, such that $\phi(A) \subset A' \subset A$. By the preceding result, we know that $v^* \in A$. Now, since $\phi(v^*) = v^*$ and $\phi(A) \subset A'$, it follows that $v^* \in A'$.

Q.E.D.

Monotonicity

One property that might be useful to have is monotonicity in the state variable, that is, we might want the value function to increase in the values x of the state variable. To this end, we make the following assumptions:

B5 X is a Borel measurable subspace of some euclidean space $\mathbb{R}^k$, and for any $z \in Z$,

$$\Gamma(x, z) \subset \Gamma(x', z)$$

for any $x, x' \in X$ such that $x \leq x'$.

Furthermore, for any $x, x' \in X$ such that $x \leq x'$ but $x \neq x'$, and $y \in \Gamma(x, z)$, we have

$$U(x, y, z) < U(x', y, z),$$

where the second term is well-defined because $y \in \Gamma(x', z)$ by the monotonicity of Γ.

B5' Z is a Borel measurable subspace of some euclidean space $\mathbb{R}^k$, and for any $x \in X$,

$$\Gamma(x, z) \subset \Gamma(x, z')$$

whenever $(z, z') \in Z$ and $z \leq z'$.

Furthemrore, for any $z, z' \in Z$ such that $z \leq z'$ but $z \neq z'$, and $(x, y) \in X \times X$ such that $y \in \Gamma(x, z)$, we have

$$U(x, y, z) < U(x, y, z'),$$

where the second term is well-defined because $y \in \Gamma(x', z)$ by the monotonicity of Γ.

The following result shows that, under assumptions B1 to B5 (B5'), the value function is increasing in the state x (the shock z).

Theorem (Monotonicity of the Value Function)
Suppose assumptions B1 to B5 hold, and let $V^* \in C_b(S, \mathbb{R})$ be the unique continuous and bounded solution to the functional equation. Then, for any $z \in Z$ and $x, x' \in X$ such that $x \leq x'$ and $x \neq x'$, we have $V^*(x, z) < V^*(x', z)$.

On the other hand, suppose assumptions B1 to B4 and assumption B5' hold. Letting $V^* \in C_b(S, \mathbb{R})$ be the unique continuous and bounded solution to the functional equation, for any $x \in X$ and $z, z' \in Z$ such that $z \leq z'$ and $z \neq z'$, we have $V^*(x, z) < V^*(x, z')$.

Proof) **Monotonicity in the State**

Suppose that assumptions B1 to B5 hold true. Let C_1 be the subset of $C_b(S, \mathbb{R})$ that collects functions that are monotonically increasing on X; formally,

$$C_1 = \{f \in C_b(S, \mathbb{R}) \mid f(x, z) \leq f(x', z) \quad \forall z \in Z, \quad x, x' \in X \quad \text{s.t.} \quad x \leq x'\}.$$

Note that C_1 is closed under the supremum norm; to see this, let $\{f_n\}_{n \in N_+}$ be a sequence in C_1 converging to some $f \in C_b(S, \mathbb{R})$ in the metric d_∞ (converging uniformly to f). For any $z \in Z$ and $x, x' \in X$ such that $x \leq x'$, for any $n \in N_+$ we have $f_n(x, z) \leq f_n(x', z)$. Since uniform convergence implies pointwise convergence, taking $n \to \infty$ on both sides shows us that $f(x, z) \leq f(x', z)$. This holds for any $z \in Z$ and $x, x' \in X$ such that $x \leq x'$, so by definition $f \in C_1$, which establishes the closedness of C_1.

Now define C_2 as the set of all functions on S that are strictly increasing in its first k arguments. Clearly, $C_2 \subset C_1$.

Letting $T : C_b(S, \mathbb{R}) \to C_b(S, \mathbb{R})$ be the contraction mapping defined as

$$[T(f)](x, z) = \sup_{y \in \Gamma(x,z)} \left[U(x, y, z) + \beta \cdot \int_Z f(y, z') P(z, dz') \right]$$

for any $f \in C_b(S, \mathbb{R})$ and $(x, z) \in S$, we can show that $T(C_1) \subset C_2$. To see this, let $f \in C_1$, and choose any $z \in Z$ and $x, x' \in X$ such that $x \leq x'$ but $x \neq x'$. From the compactness of $\Gamma(x, z)$ and the continuity of the mapping

$$y \mapsto U(x, y, z) + \beta \cdot \int_Z f(y, z') P(z, dz'),$$

the extreme value theorem implies that there exists some $y^* \in \Gamma(x,z)$ such that

$$[T(f)](x,z) = U(x,y^*,z) + \beta \cdot \int_Z f(y^*,z')P(z,dz').$$

By assumption, $U(x,y^*,z) < U(x',y^*,z)$, so we have

$$U(x,y^*,z) + \beta \cdot \int_Z f(y^*,z')P(z,dz') < U(x',y^*,z) + \beta \cdot \int_Z f(y^*,z')P(z,dz').$$

Furthermore, $y^* \in \Gamma(x,z) \subset \Gamma(x',z)$, so we can see that

$$\begin{aligned}[T(f)](x,z) &= U(x,y^*,z) + \beta \cdot \int_Z f(y^*,z')P(z,dz') \\ &< \sup_{y \in \Gamma(x',z)} \left[U(x',y,z) + \beta \cdot \int_Z f(y,z')P(z,dz') \right] = [T(f)](x',z).\end{aligned}$$

This demonstrates that $T(f)$ is a continuous and bounded function on S that is strictly increasing in its first k arguments, so that $T(f) \in C_2$.

Since V^* is the unique fixed point of the contraction mapping T, by the properties theorem $V^* \in C_2$, so that

$$V^*(x,z) < V^*(x',z) \quad \text{for any } z \in Z, x, x' \in X \quad \text{s.t.} \quad x \leq x', x \neq x'.$$

Monotonicity in the Shock

Now suppose that assumptions B1 to B4 and assumption B5' hold. As above, define C_1 to be the subset of $C_b(S,\mathbb{R})$ of functions that are increasing in the shock:

$$C_1 = \{f \in C_b(S,\mathbb{R}) \mid f(x,z) \leq f(x,z') \quad \forall x \in X,\ z,z' \in Z \text{ s.t. } z \leq z'\}.$$

Once again, C_1 is closed in the supremum norm; for any sequence $\{f_n\}_{n \in N_+}$ of functions in C_1 converging uniformly to $f \in C_b(S,\mathbb{R})$, note that, for any $x \in X$ and $z,z' \in Z$ such that $z \leq z'$,

$$f_n(x,z) \leq f_n(x,z') \quad \text{for any } n \in N_+.$$

Since uniform convergence implies pointwise convergence, taking $n \to \infty$ on both sides implies

$$f(x,z) \leq f(x,z'),$$

and because this holds for any $x \in X$ and $z, z' \in Z$ such that $z \leq z'$, by definition $f \in C_1$.

Letting C_2 be the set of functions in $C_b(S, \mathbb{R})$ that are strictly increasing in their last k arguments, we can clearly see that $C_2 \subset C_1$.

Now we will show that $T(C_1) \subset C_2$, where $T : C_b(S, \mathbb{R}) \to C_b(S, \mathbb{R})$ is the operator defined above. Choose any $f \in C_1$ and $x \in X$, $z, z' \in Z$ such that $z \leq z'$ but $z \neq z'$. Then,

$$\begin{aligned}[T(f)](x,z) &= \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \int_Z f(y,z_0) P(z, dz_0) \right] \\ &= U(x,y^*,z) + \beta \int_Z f(y,^* z_0) P(z, dz_0)\end{aligned}$$

for some $y^* \in \Gamma(x,z)$, where we use the same extreme value theorem argument as above. Now note that $U(x,y^*,z) < U(x,y^*,z')$ and that

$$\int_Z f(y^*, z_0) P(z, dz_0) \leq \int_Z f(y,^* z_0) P(z', dz_0)$$

because $f(y^*, \cdot)$ is an increasing function on Z and P is a monotonic kernel ($P(z', \cdot)$ first order stochastically dominates $P(z, \cdot)$). Therefore,

$$U(x,y^*,z) + \beta \int_Z f(y,^* z_0) P(z, dz_0) < U(x,y^*,z') + \beta \int_Z f(y,^* z_0) P(z', dz_0),$$

and because $y^* \in \Gamma(x,z) \subset \Gamma(x,z')$, we have

$$\begin{aligned}[T(f)](x,z) &= U(x,y^*,z) + \beta \int_Z f(y,^* z_0) P(z, dz_0) \\ &< \sup_{y \in \Gamma(x,z')} \left[U(x,y,z') + \beta \int_Z f(y,z_0) P(z', dz_0) \right] = [T(f)](x,z').\end{aligned}$$

This proves that $T(f) \in C_2$.

Since V^* is the unique fixed point of the contraction mapping T, by the properties theorem $V^* \in C_2$, so that

$$V^*(x,z) < V^*(x,z') \quad \text{for any } x \in X,\ z,z' \in Z \quad \text{s.t.} \quad z \leq z', z \neq z'.$$

Q.E.D.

Concavity and Single-Valuedness

Now we want to find assumptions under which we can establish concavity of the value function and the single-valuedness (=uniqueness) of the policy correspondence. For starters, we can return back to the deterministic case. There, we saw that a set of sufficient conditions for the optimal plan to be unique was for the instantaneous return function U to be strictly concave in both the current and future states, as well as for the constraint to be convex in the sense defined there. It turns out that these two conditions suffice for the single-valuedness of the policy correspondence in our stochastic case as well. Formally, we make the following assumption:

B6 X is a convex set, and for any $z \in Z$, $\Gamma(\cdot, z)$ is convex in the following sense: for any $\lambda \in [0,1]$, $x, x' \in X$ and $y \in \Gamma(x,z)$, $y' \in \Gamma(x',z)$, we have

$$\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x', z).$$

Furthermore, $U(\cdot,\cdot,z)$ is concave on the set

$$\mathrm{Gr}(\Gamma; z) = \{(x,y) \in X \times X \mid y \in \Gamma(x,z)\}.$$

for any $z \in Z$, and for any $(x,z) \in S$, $U(x,\cdot,z)$ is strictly concave on $\Gamma(x,z)$.

Note that the restriction on Γ imposed above shows us that each Γ is a convex-valued correspondence; to see this, choose any $(x,z) \in S$ and let $y, y' \in \Gamma(x,z)$ and $\lambda \in [0,1]$. Then, because $y, y' \in \Gamma(x,z)$,

$$\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x, z) = \Gamma(x,z),$$

which shows us that $\Gamma(x,z)$ is a convex set. Thus, the assumption on Γ above can be seen as a stronger version of the assumption that Γ is convex-valued.

The next theorem establishes that assumptions B1 to B4 along with B6 are sufficient for the value function to be strictly concave on X, and for the policy correspondence to be single-valued.

Theorem (Concavity of the Value Function)
Suppose assumption B6 holds alongside assumptions B1 to B4, and let $V^* \in C_b(S, \mathbb{R})$ be the unique continuous and bounded solution to the functional equation. Then, for any $z \in Z$, $V^*(\cdot, z)$ is concave on X, and the policy correspondence G^* is single-valued. By implication, there exists a continuous policy function $g^* : S \to X$ satisfying

$$V^*(x,z) = U(x, g^*(x,z), z) + \beta \cdot \int_Z V^*(g^*(x,z), z') P(z, dz')$$

for any $(x, z) \in S$.

Proof) Let C_1 and C_2 be defined as follows:

$$C_1 = \{f \in C_b(S, \mathbb{R}) \mid f(\cdot, z) \quad \text{is concave } \forall z \in Z\}.$$

C_1 is a closed subset of $C_b(S, \mathbb{R})$ under the supremum norm. Let $\{f_n\}_{n \in N_+}$ be a subset of C_1 that converges uniformly to some $f \in C_b(S, \mathbb{R})$. For any $z \in Z$, $\lambda \in [0, 1]$ and $x, x' \in X$, by definition

$$f_n(\lambda x + (1-\lambda)x', z) \geq \lambda \cdot f_n(x, z) + (1-\lambda) \cdot f_n(x', z)$$

for any $n \in N_+$, so taking $n \to \infty$ on both sides yields

$$f(\lambda x + (1-\lambda)x', z) \geq \lambda \cdot f(x, z) + (1-\lambda) \cdot f(x', z).$$

This holds for any $z \in Z$, $\lambda \in [0, 1]$ and $x, x' \in X$, so it follows that $f \in C_1$.

Next, we show that $T(C_1) \subset C_1$. Fix $f \in C_1$, and choose $z \in Z$, $\lambda \in (0, 1)$, and distinct $x, x' \in X$. Then, choose $y \in \Gamma(x, z)$ and $y' \in \Gamma(x', z)$ so that

$$[T(f)](x,z) = U(x, y, z) + \beta \cdot \int_Z f(y, z') P(z, dz')$$

$$[T(f)](x',z) = U(x', y', z) + \beta \cdot \int_Z f(y', z') P(z, dz'),$$

where the extreme value theorem allows us to make these choices in light of the fact that $[T(f)](x, z)$ and $[T(f)](x', z)$ are the maxima of continuous functions over the compact sets $\Gamma(x, z)$ and $\Gamma(x', z)$. Note also that, by the convexity of

the constraint correspondence,

$$\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x', z)$$

and as such $(x,y),(x',y'),\lambda(x,y)+(1-\lambda)(x',y') \in \mathrm{Gr}(\Gamma;z)$.

By the concavity of $U(\cdot,\cdot,z)$ on $\mathrm{Gr}(\Gamma;z)$ and the concavity of f on X,

$$\begin{aligned}
&\lambda\cdot[T(f)](x,z)+(1-\lambda)\cdot[T(f)](x',z)\\
&=\lambda\left[U(x,y,z)+\beta\cdot\int_Z f(y,z')P(z,dz')\right]+(1-\lambda)\left[U(x',y',z)+\beta\cdot\int_Z f(y',z')P(z,dz')\right]\\
&\le U(\lambda(x,y)+(1-\lambda)(x',y'),z)+\beta\cdot\int_Z\left[\lambda\cdot f(y,z')+(1-\lambda)f(y',z')\right]P(z,dz')\\
&\le U(\lambda(x,y)+(1-\lambda)(x',y'),z)+\beta\cdot\int_Z f(\lambda y+(1-\lambda)y',z')P(z,dz').
\end{aligned}$$

Since $\lambda y+(1-\lambda)y' \in \Gamma(\lambda x+(1-\lambda)x',z)$, we have

$$\begin{aligned}
&\lambda\cdot[T(f)](x,z)+(1-\lambda)\cdot[T(f)](x',z)\\
&\le \sup_{y\in\Gamma(\lambda x+(1-\lambda)x',z)}\left[U(\lambda x+(1-\lambda)x',y,z)+\beta\cdot\int_Z f(y,z')P(z,dz')\right]\\
&= [T(f)](\lambda x+(1-\lambda)x',z).
\end{aligned}$$

This holds for any $z\in Z$, $\lambda\in(0,1)$ and distinct $x,x'\in X$, so by definition $T(f)$ is strictly concave on X, and $T(f)\in C_1$.

We have shown that $T(C_1)\subset C_1$, where C_1 is a closed subset of $C_b(S,\mathbb{R})$. V^* is the unique fixed point of T on $C_b(S,\mathbb{R})$, so by the properties theorem, $V^*\in C_1$, that is, $V^*(\cdot,z)$ is concave on X for any $z\in Z$.

It remains to establish the claim about the single-valuedness of G^*. For any $(x,z)\in S$, since G^* is non-empty valued, $G^*(x,z)\neq\emptyset$. Suppose that there exist

$y, y' \in G^*(x,z)$ such that $y \neq y'$. By definition,

$$\begin{aligned} V^*(x,z) &= U(x,y,z) + \beta \cdot \int_Z V^*(y,z')P(z,dz') \\ &= U(x,y',z) + \beta \cdot \int_Z V^*(y',z')P(z,dz'). \end{aligned}$$

Putting $\lambda = \frac{1}{2} \in (0,1)$, it now follows that

$$\begin{aligned} V^*(x,z) &= \lambda \left[U(x,y,z) + \beta \cdot \int_Z V^*(y,z')P(z,dz') \right] + (1-\lambda) \left[U(x,y',z) + \beta \cdot \int_Z V^*(y',z')P(z,dz') \right] \\ &< U(x, \lambda y + (1-\lambda)y', z) + \beta \cdot \int_Z \left[\lambda \cdot V^*(y,z') + (1-\lambda) \cdot V^*(y',z') \right] P(z,dz') \\ &\leq U(x, \lambda y + (1-\lambda)y', z) + \beta \cdot \int_Z V^*(\lambda y + (1-\lambda)y', z')P(z,dz'), \end{aligned}$$

where we used, in quick succession, the strict concavity of $U(x,\cdot,z)$ on X and the concavity of V^* on X. Since $\lambda y + (1-\lambda)y' \in \Gamma(x,z)$ because $y, y' \in \Gamma(x,z)$ and $\Gamma(\cdot,z)$ is convex, the above contradicts the fact that V^* solves the functional equation. Therefore, we must have $y = y'$, and $G^*(x,z)$ is a singleton. This holds for any $(x,z) \in S$, so G^* is single valued and V^* admits an associated policy function g^*.

Q.E.D.

4.3.3 Iterative Method for Finding Value and Policy Functions

We take a brief detour, now that we have established the existence of a policy function under the appropriate assumptions, to furnish the theoretical foundation for a method that allows us to find both the policy and value functions by iterative methods. The formal statement is given below; throughout, let $T : C_b(S,\mathbb{R}) \to C_b(S,\mathbb{R})$ be the contraction mapping defined as

$$[T(f)](x,z) = \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz') \right]$$

for any $f \in C_b(S,\mathbb{R})$ and $(x,z) \in S$. In addition, let C_1, C_2 be defined as in the above theorem, namely as the set of all bounded and continuous functions on $X \times Z$ that are concave and strictly concave on X, respectively.

Theorem (Pointwise Convergence of Policy Functions)
Suppose assumption B6 holds alongside assumptions B1 to B4. Let $V^* \in C_b(S,\mathbb{R})$ be the unique continuous and bounded solution to the functional equation, and $g^* : S \to X$ the corresponding policy function. choose any function $V^{(0)} \in C_1$. Define the sequence $\{V^{(n)}\}_{n\in\mathbb{N}}$ of continuous bounded real functions on S and $\{g_n\}_{n\in\mathbb{N}}$ of functions from S to X as

$$V^{(n+1)} = T(V^{(n)})$$

$$g_n(x,z) = \operatorname*{argmax}_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V^{(n)}(y,z') P(z,dz') \right]$$

for any $(X,z) \in S$ and $n \in \mathbb{N}$. Then, $\{V^{(n)}\}_{n\in\mathbb{N}}$ converges uniformly to V^*, and $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise to g^*.

Proof) Define the operator $L : C_b(S,\mathbb{R}) \to C_b(X \times X \times Z, \mathbb{R})$ as

$$[L(f)](x,y,z) = U(x,y,z) + \beta \cdot \int_Z f(y,z') P(z,dz')$$

for any $f \in C_b(S,\mathbb{R})$ and $(x,y,z) \in \mathrm{Gr}(\Gamma)$. We proceed in steps:

Step 1: The Sequences are Well-defined

We first check that the sequences above are well-defined. First, we saw in the proof of existence that $\{V^{(n)}\}_{n\in\mathbb{N}}$ converges uniformly, that is, in the supremum norm, to V^*. Furthermore, since we saw that $T(C_1) \subset C_2 \subset C_1$ and $V^{(0)} \in C_1$, it follows from induction that $\{V^{(i)}\}_{i\in N_+} \subset C_2$. Fixing any $i \in \mathbb{N}$, define $G^{(i)} : S \to X$ as

$$G^{(i)}(x,z) = \{y \in \Gamma(x,z) \mid V^{(i+1)}(x,z) = [L(V^{(i)})](x,y,z)\}$$

for any $(x,z) \in S$. Then, since $G^{(i)}$ is the set of maximizers for the maximization problem

$$\max_{y\in\Gamma(x,z)} \quad [L(V^{(i)})](x,y,z),$$

by Berge's theorem $G^{(i)}$ is non-empty compact valued and upper hemicontinuous. As such, for any $(x,z) \in S$, $G^{(i)}(x,z) \neq \emptyset$. Suppose $y, y' \in G^{(i)}(x,z)$ such that $y \neq y'$. Putting $\lambda = \frac{1}{2} \in (0,1)$, since

$$V^{(i+1)}(x,z) = [L(V^{(i)})](x,y,z) = [L(V^{(i)})](x,y',z),$$

we can see that

$$\begin{aligned}
V^{(i+1)}(x,z) &= \lambda \cdot [L(V^{(i)})](x,y,z) + (1-\lambda) \cdot [L(V^{(i)})](x,y',z) \\
&= \lambda \cdot \left[U(x,y,z) + \beta \cdot \int_Z V^{(i)}(y,z')P(z,dz')\right] \\
&\quad + (1-\lambda) \cdot \left[U(x,y',z) + \beta \cdot \int_Z V^{(i)}(y',z')P(z,dz')\right] \\
&< U(x,\lambda y + (1-\lambda)y',z) + \beta \cdot \int_Z V^{(i)}\left(\lambda y + (1-\lambda)y', z'\right) P(z,dz') \\
&= [L(V^{(i)})](x,\lambda y + (1-\lambda)y',z),
\end{aligned}$$

where the inequality follows from the strict concavity of $U(x,\cdot,z)$ on $\Gamma(x,z)$ and

the concavity of $V^{(i)}$ on X (each $V^{(i)} \in C_1$). Since $\lambda y + (1-\lambda)y' \in \Gamma(x,z)$, we have a contradiction, so that $y = y'$ and $G^{(i)}(x,z)$ is a singleton. This indicates that there exists a policy function g_i such that

$$g_i(x,z) = \underset{y \in \Gamma(x,z)}{\operatorname{argmax}} \left[U(x,y,z) + \beta \cdot \int_Z V^{(i)}(y,z') P(z,dz') \right]$$

for any $(x,z) \in S$. This holds for any $i \in \mathbb{N}$, so it follows that the sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ is well-defined.

Step 2: Preliminary Result

To show that $\{g_n\}_{n \in \mathbb{N}}$ converges pointwise to g^*, we first establish a preliminary result. Fixing $(x,z) \in S$, choose any $\epsilon > 0$. We want to show that there exists a $\delta(x,z) > 0$ such that

$$y \in \Gamma(x,z) \quad \text{and} \quad |V^*(x,z) - [L(V^*)](x,y,z)| < \delta(x,z)$$

implies $|g^*(x,z) - y| < \epsilon$.

Define the set

$$A = \{y \in \Gamma(x,z) \mid |g^*(x,z) - y| \geq \epsilon\}.$$

If $A = \emptyset$, then we can set $\delta(x,z)$ equal to any positive value, since $|g^*(x,z) - y| < \epsilon$ for any $y \in \Gamma(x,z)$.

Now suppose that $A \neq \emptyset$, and let

$$y^* = \underset{y \in A}{\operatorname{argmax}} \, [L(V^*)](x,y,z),$$

where $y^* \in \Gamma(x,z)$ is well-defined because the objective function is continuous and A, being the intersection of a closed set and a compact set $\Gamma(x,z)$, is itself

compact. We define

$$\delta(x,z) = \underbrace{V^*(x,z)}_{[L(V^*)](x,g^*(x,z),z)} - [L(V^*)](x,y^*,z) > 0,$$

where the inequality is strict because $g^*(x,z)$ is the unique maximizer of $[L(V^*)](x,y,z)$ on $\Gamma(x,z)$ and $g^*(x,z) \notin A$. Choose any $y \in \Gamma(x,z)$ such that

$$\begin{aligned} |V^*(x,z) - [L(V^*)](x,y,z)| &= [L(V^*)](x,g^*(x,z),z) - [L(V^*)](x,y,z) \\ &< \delta(x,z). \end{aligned}$$

Then, we have

$$[L(V^*)](x,y^*,z) < [L(V^*)](x,y,z),$$

which implies that $y \notin A$, since y^* is the maximizer of $[L(V^*)](x,y,z)$ on $\Gamma(x,z)$. Therefore, $|g^*(x,z) - y| < \epsilon$, which completes the proof of this preliminary result.

Step 3: Pointwise Convergence of Policy Functions

Now fix $(x,z) \in S$ and choose any $\epsilon > 0$. We want to show that there exists an $N \in N_+$ such that

$$|g_n(x,z) - g^*(x,z)| < \epsilon$$

for any $n \geq N$. First, the preceding result shows us that there exists some $\delta(x,z)$ such that, for any $y \in \Gamma(x,z)$ such that

$$V^*(x,z) - [L(V^*)](x,y,z) < \delta(x,z),$$

we have $|y - g^*(x,z)| < \epsilon$. It follows that we need only find an $N \in N_+$ such that

$$V^*(x,z) - [L(V^*)](x,g_n(x,z),z) < \delta(x,z)$$

for any $n \geq N$.

Returning to the definition of each g_n, since

$$[L(V^{(n)})](x, g^*(x,z), z) \leq [L(V^{(n)})](x, g_n(x,z), z) = V^{(n+1)}(x,z),$$

for any $n \in N_+$, we have

$$\begin{aligned}
&0 \leq V^*(x,z) - [L(V^*)](x, g_n(x,z), z) \\
&= [L(V^*)](x, g^*(x,z), z) - [L(V^{(n)})](x, g^*(x,z), z) + [L(V^{(n)})](x, g^*(x,z), z) - [L(V^*)](x, g_n(x,z), z) \\
&\leq [L(V^*)](x, g^*(x,z), z) - [L(V^{(n)})](x, g^*(x,z), z) + [L(V^{(n)})](x, g^n(x,z), z) - [L(V^*)](x, g_n(x,z), z) \\
&\leq 2 \cdot \sup_{y \in \Gamma(x,z)} \Big| [L(V^*)](x,y,z) - [L(V^{(n)})](x,y,z) \Big|.
\end{aligned}$$

For any $y \in \Gamma(x,z)$ and $n \in N_+$,

$$\begin{aligned}
\Big| [L(V^*)](x,y,z) - [L(V^{(n)})](x,yz) \Big| &= \left| \beta \cdot \int_Z \left(V^*(y,z') - V^{(n)}(y,z') \right) P(z, dz') \right| \\
&\leq \beta \cdot \int_Z \Big| V^*(y,z') - V^{(n)}(y,z') \Big| P(z, dz') \\
&\leq \beta \Big\| V^* - V^{(n)} \Big\|_\infty,
\end{aligned}$$

so putting the results together,

$$0 \leq V^*(x,z) - [L(V^*)](x, g_n(x,z), z) \leq 2\beta \Big\| V^* - V^{(n)} \Big\|_\infty$$

for any $n \in N_+$. Since the right hand side goes to 0 as $n \to \infty$, we can choose $N \in N_+$ so that

$$2\beta \Big\| V^* - V^{(n)} \Big\|_\infty < \delta(x,z)$$

for any $n \geq N$; it follows then that

$$V^*(x,z) - [L(V^*)](x, g_n(x,z), z) < \delta(x,z)$$

and therefore

$$|g_n(x,z) - g^*(x,z)| < \epsilon$$

for any $n \geq N$. This holds for any $(x,z) \in S$, so $\{g_n\}_{n \in N_+}$ converges to g^* pointwise.

Q.E.D.

Let us think for a moment about what the above theorem tells us. It states that, given that the initial value function $V^{(0)}$ is chosen to be concave in the state variable, we can iteratively search for the value function and the policy function by constructing $V^{(i+1)}$ as $T(V^{(i)})$ given the ith iterate $V^{(i)}$, and by letting $g^{(i)} : S \to X$ be the function that maximizes, for each value of $(x,z) \in S$, the objective function

$$U(x,y,z) + \beta \cdot \int_Z V^{(i)}(y,z') P(z,dz')$$

over $\Gamma(x,z)$.

4.3.4 Differentiability

Another property that we want to establish is the differentiability of the value function. This would allow for us to obtain first order conditions for maximization, using which we can obtain the Euler equation, as in the sequential approach to solving deterministic dynamic problems. It turns out that a sufficient condition for the value function to be differentiable is just for the instantaneous return function to be differentiable at some point on the interior of X:

B7 X is a convex subset of some euclidean space $\mathbb{R}^k$ with non-empty interior. Furthermore, for any $z \in Z$ the function $U(\cdot, \cdot, z)$ is continuously differentiable on the interior of $\text{Gr}(\Gamma; z)$.

Under assumptions B1 to B4 and assumptions B6, B7, the unique continuous and bounded solution V^* to the functional equation turns out to be continuously differentiable on the interior of the state space X. First, we prove the following result, which plays a key part in our proof of the differentiability of V^*:

Lemma (Benveniste-Scheinkman Lemma)
Let E be an open convex subset of $\mathbb{R}^k$, and $f : E \to \mathbb{R}$ a convex function. Suppose there exists a convex function $g : E \to \mathbb{R}$ that is differentiable on E such that $g(x) \geq f(x)$ for any $x \in E$ and $g(x_0) = f(x_0)$ for some $x_0 \in E$. Then, f is differentiable at x_0 with derivative given as

$$\nabla f(x_0) = \nabla g(x_0).$$

Proof) For any $x \in E$, let

$$\partial f(x) = \{v \in \mathbb{R}^k \mid f(z) \geq f(x) + v'(z - x) \quad \forall z \in E\}$$

be the subdifferential of f at x. Recall (from the text on convex analysis and optimization) that $\partial f(x)$ is non-empty for any $x \in E$ and f is differentiable at some $x \in E$ if and only if $\partial f(x)$ is a singleton, in which case the gradient of f at x equals the single element of $\partial f(x)$. We use this fact to prove the theorem.

Since g is convex and differentiable on E, by the characterization above $\partial g(x) =$

$\{\nabla g(x)\}$ for any $x \in E$. For any $v \in \partial f(x_0)$, by definition

$$f(z) \geq f(x_0) + v'(z - x_0)$$

for any $z \in E$. Since $g(z) \geq f(z)$ for any $z \in E$ and $f(x_0) = g(x_0)$, we can see that

$$v'(z - x_0) + g(x_0) = v'(z - x_0) + f(x_0) \leq f(z) \leq g(z)$$

for any $z \in E$, so that $v \in \partial g(x_0)$. This implies that

$$\partial f(x_0) \subset \partial g(x_0) = \{\nabla g(x_0)\},$$

so that $\partial f(x_0) = \{\nabla g(x_0)\}$. As such, f is differentiable at x_0 with derivative given as

$$\nabla f(x_0) = \nabla g(x_0).$$

Q.E.D.

To establish the desired differentiability result, we require another lemma, this time about the behavior of lower hemicontinuous and convex-valued correspondences.

Lemma Let (E, d) be an arbitrary metric space, and F a subset of $\mathbb{R}^m$. Suppose that $\Phi : E \to F$ is a non-empty convex valued and lower hemicontinuous correspondence. In this case, if $y_0 \in \Phi(x_0)^o$ for some $(x_0, y_0) \in E \times F$, then there exists a neighborhood V of x_0 such that $y_0 \in \Phi(x)$ for any $x \in V$.

Proof) The geometric intuition for the proof is simple. If there does not exist such a neighborhood V of x_0, then by the separating hyperplane theorem the convex sets $\Phi(x_n)$ and the singleton $\{y_0\}$ will always be separated by a sequence of hyperplanes. But becauase y_0 is in the interior of $\Phi(x_0)$ and Φ is lower hemicontinuous, we can simply choose a point in $\Phi(x_0)$ in the opposite direction of the constructed hyperplanes and find a sequence that converges to this point but

whose elements are contained in the sets $\Phi(x_n)$. Eventually, when n becomes large enough, the elements of this sequence, which belong to $\Phi(x_n)$, will cross over to the "wrong" side of the hyerplane, resulting in a contradiction.

We now state the formal proof of this result. Since y_0 is contained in the interior of $\Phi(x_0)$, there exists an $\epsilon > 0$ such that $B(y_0, \epsilon) \subset \Phi(x_0)$. Suppose that the claim of the lemma is false; then, for any $n \in N_+$, there exists some $x_n \in E$ such that $d(x_n, x) < \frac{1}{n}$ and $y_0 \notin \Phi(x_n)$. By design, $\{x_n\}_{n \in N_+}$ converges to x. Each $\Phi(x_n)$ is a non-empty convex set such that $y_0 \notin \Phi(x_n)$, so by the separating hyperplane theorem, there exists some non-zero vector $v_n \in \mathbb{R}^m$ such that

$$v_n'(y - y_0) \geq 0$$

for any $y \in \Phi(x_n)$; we normalize the norm of v_n to 1.

Since the sequence $\{v_n\}_{n \in N_+}$ takes values in the unit circle $\mathbb{T}^m = \{z \in \mathbb{R}^m \mid |z| = 1\}$ on $\mathbb{R}^m$, which is a bounded set, by the Bolzano-Weierstrass theorem $\{v_n\}_{n \in N_+}$ has a convergent subsequence. To save on notation, without loss of generality we let $\{v_n\}_{n \in N_+}$ itself be this subsequence, and denote its limit by $v^* \in \mathbb{T}^m$.

Now define $z = y_0 - \frac{\epsilon}{2} \cdot v \in \Phi(x_0)$. Since $|z - y_0| = \frac{\epsilon}{2} \cdot |v| = \frac{\epsilon}{2} < \epsilon$, $z \in \Phi(x_0)$. By lower hemicontinuity there exists a subsequence $\{x_{n_k}\}_{k \in N_+}$ of $\{x_n\}_{n \in N_+}$ and a sequence $\{y_k\}_{k \in N_+}$ in F such that $y_k \in \Gamma(x_{n_k})$ for any $k \in N_+$ and $y_k \to z$ as $k \to \infty$. Therefore, for any $k \in N_+$,

$$v_{n_k}'(y_k - y_0) \geq 0$$

so that

$$v_{n_k}'(y_k - z) = v_{n_k}'(y_k - y_0) + v_{n_k}'(y_0 - z) \geq v_{n_k}'(y_0 - z) = \frac{\epsilon}{2}(v_{n_k}' v).$$

Taking $k \to \infty$ on both sides yields

$$0 = \lim_{k \to \infty} v_{n_k}'(y_k - z) \geq \frac{\epsilon}{2}(v'v) = \frac{\epsilon}{2},$$

a contradiction. Therefore, there must exist some $\delta > 0$ such that $y_0 \in \Phi(x)$ for any $x \in B(x_0, \delta)$.

Q.E.D.

We can now show that the value function is differentiable under the assumptions we have established so far, alongside an additional interiority assumption on the optimal policy function.

Theorem (Differentiability of Value Function)
Suppose assumptions B6 and B7 hold alongside assumptions B1 to B4. Let $V^* \in C_b(S, \mathbb{R})$ be the unique continuous and bounded solution to the functional equation, and let $g^* : S \to X$ be the associated policy function. For some $(x_0, z) \in S$, assume that one of the following two conditions hold:

i) $((x_0, z), g^*(x_0, z))$ lies in the interior of the graph of Γ.

ii) $g^*(x_0, z)$ lies in the interior of $\Gamma(x_0, z)$, that is, $g^*(x_0, z) \in \Gamma(x_0, z)^o$.

Then, $V^*(\cdot, z)$ is differentiable at x_0 with gradient

$$\nabla_x V^*(x_0, z) = \nabla_x U(x_0, g^*(x_0, z), z).$$

Proof) From the preceding theorem, we saw that, under assumptions B1 to B4 and assumption B6, $V^*(\cdot, z)$ is strictly concave for any $z \in Z$ and the policy correspondence is single-valued, so that there is a continuous policy function $g^* : S \to X$ associated with V^*.

Choose any $z \in Z$ and $x_0 \in X^o$ so that either one of the conditions hold. Suppose $((x_0, z), g^*(x_0, z)) \in Gr(\Gamma)^o$, where $Gr(\Gamma) \subset S \times X$ is the graph of Γ defined as

$$Gr(\Gamma) = \{(s, x) \in S \times X \mid x \in \Gamma(s)\}.$$

Then, there exists an $\epsilon > 0$ such that the ϵ-ball around $((x_0, z), g^*(x_0, z))$ is contained in the graph $Gr(\Gamma)$. Then, for any $x \in X$ such that $|x - x_0| < \epsilon$,

$((x,z),g^*(x_0,z)) \in Gr(\Gamma)$, so that $g^*(x_0,z) \in \Gamma(x,z)$.

On the other hand, suppose $g^*(x_0,z) \in \Gamma(x_0,z)^o$. Since (X,d) is a subset of euclidean space, Γ is non-empty convex valued (by assumption B6) and lower hemicontinuous (by assumption B3), the preceding lemma holds. Specifically, we can see that there exists some neighborhood around $x_0 \in X$ such that $g^*(x_0,z) \in \Gamma(x,z)$ for any x contained in this neighborhood.

In any case, we can see that there exists a neighborhood N of x_0 such that $g^*(x_0,z) \in \Gamma(x,z)$ for any $x \in N$.

Now define $W : N \to \mathbb{R}$ as

$$W(x) = U(x,g^*(x_0,z),z) + \beta \cdot \int_Z V^*(g^*(x_0,z),z')P(z,dz')$$

for any $x \in N$. Then, W is a concave (assumption B6) and differentiable (assumption B7) function on the open and convex subset N of $\mathbb{R}^k$ such that

$$\begin{aligned} V^*(x,z) &= \sup_{y \in \Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V^*(y,z')P(z,dz') \right] \\ &\geq U(x,g^*(x_0,z),z) + \beta \cdot \int_Z V^*(g^*(x_0,z),z')P(z,dz') \\ &\qquad\qquad \text{(Since } g^*(x_0,z) \in \Gamma(x,z)\text{)} \\ &= W(x) \end{aligned}$$

for any $x \in N$. Furtheremore, by definition

$$V^*(x_0,z) = U(x_0,g^*(x_0,z),z) + \beta \cdot \int_Z V^*(g^*(x_0,z),z')P(z,dz') = W(x_0),$$

so that the Benveniste-Scheinkman lemma now tells us $V^*(\cdot,z)$ is differentiable at x_0 with derivative given as

$$\boldsymbol{\nabla}_x V^*(x_0,z) = \boldsymbol{\nabla}_x U(x_0,g^*(x_0,z),z).$$

Q.E.D.

4.3.5 Summary of Results

So far in this chapter, we have studied stochastic dynamic optimization problems given in the following form:

$$\begin{aligned} \max_{\{\pi_t\}_{t\in\mathbb{N}}} \quad & U(x_0,\pi_0,z_0)+\sum_{t=1}^{\infty}\beta^t\cdot\int_{Z^t}U(\pi_{t-1}(z^{t-1}),\pi_t(z^t),z_t)\mu^t(z_0,dz^t) \\ \text{subject to} \quad & \pi_0\in\Gamma(s_0) \\ & \pi_t(z^t)\in\Gamma(\pi_{t-1}(z^{t-1}),z^t) \quad \text{for any } t\in N_+, z^t\in Z^t \end{aligned}$$

given the initial value $s_0=(x_0,z_0)\in S$ of the state and the Markov shock, where $\mu^t(z_0,\cdot)$ is the distribution of the first t observations of a Markov chain with Markov kernel P initialized at $z_0\in Z$.

The functional equaion dual to the above sequence problem is given by

$$V(x,z)=\sup_{y\in\Gamma(x,z)}\left[U(x,y,z)+\beta\cdot\int_Z V(y,z')P(z,dz')\right]$$

for any $(x,z)\in S$.

We made the following assumptions, under which both the sequence problem and functional equations are well-posed (since they imply assumptions A1 and A2):

B1 : **State Space and Space of Stochastic Shocks**
The state space and space of stochastic shocks are given as (X,d) and (Z,ρ), separable and complete metric spaces.

B2 : **Feller Property of Markov Kernel** P
P has the Feller property, that is, the mapping

$$z\mapsto P(z,\cdot)$$

is a continuous function from Z into $\mathcal{M}_p(\mathcal{Z})$ with respect to the metric ρ on Z and the Prohorov metirc $\mathcal{P}$ on $\mathcal{M}_p(\mathcal{Z})$. Equivalently, for any convergent sequence

$\{z_n\}_{n\in N_+}$ in Z with limit z,

$$P(z_n,\cdot)\to P(z,\cdot)\quad \text{weakly as } n\to\infty.$$

B3 : **Well-behaved Constraint Correspondence**

The constraint correspondence $\Gamma : X\times Z\to X$ is a non-empty compact valued and continuous correspondence.

B4 : **Well-behaved Instantaneous Return Function**

The instantaneous return function $U : \mathrm{Gr}(\Gamma)\to\mathbb{R}$ is bounded and continuous on $\mathrm{Gr}(\Gamma)$.

B6 : **Concavity Assumptions**

X is a convex set, $U(\cdot,\cdot,z)$ is concave on $\mathrm{Gr}(\Gamma;z)$ for any $z\in Z$, and $U(x,\cdot,z)$ is strictly concave on $\Gamma(x,z)$ for any $(x,z)\in S$.

Furthermore, for any $z\in Z$, $\Gamma(\cdot,z)$ is convex in the sense that, for any $\lambda\in[0,1]$, $x,x'\in X$ and $y\in\Gamma(x,z), y'\in\Gamma(x',z)$,

$$\lambda y+(1-\lambda)y'\in\Gamma(\lambda x+(1-\lambda)x',z).$$

B7 : **Differentiability Assumptions**

X has a non-empty interior, and for any $z\in Z$, the function $U(\cdot,\cdot,z)$ is continuously differentiable on the interior of $\mathrm{Gr}(\Gamma;z)$.

Under the above assumptions, we showed finally that the following results hold, which demonstrate how the recursive formluation of the problem captured in the functional equation can help solve the sequence problem, the original object of interest.

1) **Existence and Uniqueness of Solution to Functional Equation**
There exists a unique solution V^* in $C_b(S,\mathbb{R})$, the space of all bounded and continuous real valued functions on S, to the functional equation:

$$V^*(x,z) = \sup_{y\in\Gamma(x,z)} \left[U(x,y,z) + \beta \cdot \int_Z V^*(y,z')P(z',dz)\right]$$

for any $(x,z) \in S$. In addition, $V^*(\cdot,z)$ is concave on X for any $z \in Z$.

2) **Continuous and Single-valued Policy Correspondence**
There exists a continuous policy function $g^* : S \to X$ such that

$$V^*(x,z) = U(x,g^*(x,z),z) + \beta \cdot \int_Z V^*(g^*(x,z),z')P(z',dz)$$

for any $(x,z) \in S$.

3) **Exhaustiveness of Policy Funciton for Optimal Plans**
Since V^* is a measurable function solving the functional equation that satisfies the transversality condition and G^* is the associated policy correspondence that is non-empty valued and admits a measurable selection (it has a policy function representation), by the first part of the principle of optimality V^* is the value function of the sequence problem, and by the second part any optimal plan $\pi^* \in \Pi(s_0)$ started from $s_0 \in S$ is almost surely generated by g^* in the sense that

$$\pi_0^* = g^*(s_0)$$

$$\pi_t^*(z^t) = g^*(\pi_{t-1}^*(z^{t-1}), z_t) \quad \text{for any } t \in N_+ \text{ and } \mu^t(z_0,\cdot) - \text{a.e. } z^t \in Z^t.$$

4) **Envelope Theorem for Value Function**
For any $s_0 = (x_0,z_0) \in S$ such that $x_0 \in X^o$ and $g^*(s_0) \in \Gamma(s_0)^o$, $V^*(\cdot,z_0)$ is differentiable at x_0 with gradient

$$\nabla_x V^*(x_0,z_0) = \nabla_x U(x_0,g^*(x_0,z_0),z_0).$$

Chapter 5

Recursive Approaches to the Social Planner's Problem

In this chapter, we use the machinery developed above to solve recursive macroeconomic models. The first type of models that we will focus on are general equilibrium models populated by homogeneous agents with complete markets in which there are no financial frictions and other imperfections; in short, these are models to which the first welfare theorem applies. Therefore, the equilibrium allocation in these models can be obtained by solving for the social planner's problem and abstracting from prices. We first study the stochastic neoclassical growth model of Brock and Mirman (1972), before moving onto the real business cycle model that incorporates endogenous labor supply.

5.1 The Stochastic Neoclassical Growth Model

Earlier, we studied the deterministic version of the neoclassical growth model using sequence methods. Due to the lack of any stochastic components in that model, we were able to show, among other results, that capital and consumption converge to their steady state levels. Here we look at a stochastic version of the same model, where now productivity evolves stochastically over time. Brock and Mirman (1972) show that, in this stochastic version of the model, capital stock does not converge deterministically to a fixed steady state level of capital, but instead converges in distribution to some probability measure that is independent of the initial capital stock. Thus, a version of the convergence result in deterministic growth models continue to hold even in the presence

of stochasticity.

5.1.1 Model Assumptions

The assumptions of the model are as follows:

Technologies Capital takes values in the state space $X = \mathbb{R}_+$, and productivity takes values in the compact interval $Z = [\underline{a}, \overline{a}] \subset \mathbb{R}$, with associated Borel σ-algebra $\mathcal{Z}$. We assume that productivity is a Markov chain that evolves according to the Markov kernel P. We further assume that P has the Feller property and is monotonic.

The firm produces final goods based on the production function

$$y_t = f(k_t, a_t),$$

where $f : \mathbb{R}_+ \times Z \to \mathbb{R}_+$ satisfies the usual conditions

$$f_k(\cdot, a) > 0, \quad f_k k(\cdot, a) < 0$$

for marginal productivity and the Inada conditions

$$\lim_{k \downarrow 0} f_k(k, a) = +\infty, \quad \lim_{k \to \infty} f_k(k, a) = 0$$

for any $a \in Z$. Furthermore, we assume that $f(k, \cdot)$ is strictly increasing on Z for any $k \in X$; in other words the higher productivity, the greater the output given the same amount of inputs.

The mapping

$$k \mapsto f(k, \overline{a}) + (1 - \delta)k$$

is a strictly concave function, and by the Inada conditions and the intermediate value theorem there exists some $k^* > 0$ that satisfies the first order condition

$$f_k(k^*, \overline{a}) + 1 - \delta = 0.$$

It follows now that

$$f(k,a)+(1-\delta)k \leq f(k,\overline{a})+(1-\delta)k \leq f(k^*,\overline{a})+(1-\delta)k^*$$

for any $(k,a) \in [0,+\infty) \times Z$, so defining

$$\hat{k} = f(k^*,\overline{a})+(1-\delta)k^*,$$

we can see that the capital stock in the economy cannot exceed $\hat{k}$. For this reason, w restrict the state space to $X = [0,\hat{k}]$; recall that we called $\hat{k}$ the maximum maintainable level of capital stock.

We assume that capital depreciates at a rate $\delta \in [0,1]$ each period, so that investment at time t satisfies

$$k_{t+1} = i_t + (1-\delta)k_t.$$

Finally, there is no government spending, nor is there a foreign sector, so that output each period is fully spent as either consumption or investment:

$$y_t = c_t + i_t.$$

Preferences The representative household's flow utility function is given as $u : \mathbb{R}_+ \to \mathbb{R}$. u is assumed to be twice continuously differentiable on $(0,+\infty)$ with derivatives

$$u'(\cdot) > 0, \quad u''(\cdot) < 0$$

on $\mathbb{R}$, so that u is strictly concave on $\mathbb{R}$. We also assume the Inada conditions

$$\lim_{c \downarrow 0} u'(c) = +\infty, \quad \lim_{c \to \infty} u'(c) = 0$$

hold. The discount factor is $\beta \in (0,1)$.

5.1.2 The Recursive Form of the Problem

Again, we set up the sequence problem and the dual functional equation, and then check whether assumptions B1 to B7 are satisfied to derive general results on the value and policy functions for this model.

The Sequence Problem and Recursive Formulation

The social planner's sequence problem is given as

$$\begin{aligned} \max_{\{c_t\}_{t\in\mathbb{N}}} \quad & \mathbb{E}_0\left[\sum_{t=1}^{\infty}\beta^t\cdot u(c_t)\right] \\ \text{subject to} \quad & k_{t+1}\in[0,\hat{k}], \quad c_t\geq 0 \\ & i_t = k_{t+1}-(1-\delta)k_t \\ & c_t + i_t = f(k_t,a_t) \quad \text{for any } t\in\mathbb{N}. \end{aligned}$$

We can once again reformulate the planner's problem in terms of the state k_t and the shock a_t:

$$\begin{aligned} \max_{\{k_t\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty}\beta^t\cdot\mathbb{E}_0\left[u\left[f(k_t,a_t)-k_{t+1}+(1-\delta)k_t\right]\right] \\ \text{subject to} \quad & k_{t+1}\in[0,\ f(k_t,a_t)+(1-\delta)k_t] \quad \text{for any } t\in\mathbb{N}. \end{aligned}$$

Defining the constraint correspondence $\Gamma : X\times Z\to X$ as

$$\Gamma(k,a) = [0,\ f(k,a)+(1-\delta)k]$$

for any $(k,a)\in X\times Z$, note that $\Gamma(\cdot)$ is non-empty compact convex valued. Since the mapping $(k,a)\mapsto f(k,a)+(1-\delta)k$ is continuous, it follows that Γ is a continuous correspondence as well.

Now the instantaneous return function $U : \mathrm{Gr}(\Gamma)\to\mathbb{R}$ can be defined as

$$U(k,k',a) = u\left[f(k,a)-k'+(1-\delta)k\right]$$

for any $(k,a) \in X \times Z$ and $k' \in \Gamma(k,a)$. The sequence problem can be written as

$$\max_{\pi \in \Pi(k_0,a_0)} \quad \sum_{t=1}^{\infty} \beta^t \cdot \int_{Z^t} U\left[\pi_{t-1}(a^{t-1}), \pi_t(a^t), a_t\right] \mu^t(a_0, da^t)$$

where:

- **Feasible Plans**
  $\Pi(k_0, a_0)$ is the set of all feasible plans given the initial values $(k_0, a_0) \in X \times Z$, that is, plans $\pi = \{\pi_t\}_{t \in \mathbb{N}}$ satisfying

$$\pi_0 \in \Gamma(k_0, a_0)$$

$$\pi_t(a^t) \in \Gamma(\pi_{t-1}(a^{t-1}), a_t) \quad \text{for any } t \in N_+, a^t \in Z^t.$$

- **Distribution of Markov Chain**
  For any $t \in N_+$, $\mu^t(a_0, \cdot)$ is the distribution of $A^t = (A_1, \cdots, A_t)$ conditional on $A_0 = a_0$. By the Markov nature of the productivity process, we saw above that these distributions are defined recursively as $\mu^1 = P$ and

$$\mu^t(a_0, B) = \int_{Z^{t-1}} \int_Z I_B(a^{t-1}, a_t) P(a_{t-1}, da_t) \mu^{t-1}(a_0, da^{t-1})$$

  for any $t \geq 2$, $a_0 \in Z$ and $B \in \mathcal{Z}^t$.

The functional equation corresponding to this sequence problem becomes

$$V(k,a) = \sup_{k' \in [0,\ f(k,a)+(1-\delta)k]} \left[u\left[f(k,a) + (1-\delta)k - k'\right] + \beta \int_Z V(k', a') P(a, da') \right].$$

Checking the General Assumptions

We now check whether the general assumptions hold.

B1 : **State Space and Space of Productivity Shocks**
The state space $X = [0, \hat{k}]$ is compact, as is the set Z on which the productivity shocks take values. This indicates that $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ are both complete and separable subspaces of euclidean space.

B2 : **Feller Property of Markov Kernel** P
P has the Feller property by assumption.

B3 : **Well-behaved Constraint Correspondence**
We already saw above that Γ is a non-empty compact and convex valued correspondence that is also continuous.

B4 : **Well-behaved Instantaneous Return Function**
The return function $U : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ is continuous on its domain, which is a compact set, so it is also bounded.

B6 : **Concavity Assumptions**

Fix $a \in Z$ and choose any $k_1' \in \Gamma(k_1, a)$, $k_2' \in \Gamma(k_2, a)$ and $\lambda \in [0, 1]$. By the concavity of $f(\cdot, a)$,

$$f(\lambda k_1 + (1-\lambda)k_2, a) - (1-\delta)(\lambda k_1 + (1-\lambda)k_2)$$
$$\geq \lambda [f(k_1, a) + (1-\delta)k_1] + (1-\lambda)[f(k_2, a) + (1-\delta)k_2],$$

and by design,

$$0 \leq k_i' \leq f(k_i) + (1-\delta)k_i \quad \text{for } i = 1, 2.$$

Therefore,

$$0 \leq \lambda k_1' + (1-\lambda)k_2' \leq \lambda\left[f(k_1,a)+(1-\delta)k_1\right] + (1-\lambda)\left[f(k_2,a)+(1-\delta)k_2\right]$$
$$\leq f(\lambda k_1 + (1-\lambda)k_2, a) - (1-\delta)(\lambda k_1 + (1-\lambda)k_2),$$

so that

$$\lambda k_1' + (1-\lambda)k_2' \in \Gamma\left[\lambda k_1 + (1-\lambda)k_2,\ a\right].$$

This shows us that Γ is appropriately convex in the sense that we desire.

Fix $a \in Z$, and let

$$\mathrm{Gr}(\Gamma; a) = \{(k,k') \in X^2 \mid k' \in \Gamma(k,a)\}$$

as usual. $\mathrm{Gr}(\Gamma; a)$ is a convex set by what we just showed. Choose any $(k_1,k_1'),(k_2,k_2') \in \mathrm{Gr}(\Gamma; a)$ and $\lambda \in [0,1]$. Then,

$$U(k_i,k_i',a) = u\left[f(k_i,a) - k_i' + (1-\delta)k_i\right]$$

for $i = 1,2$, and the strict concavity of $f(\cdot,a)$ implies that

$$f(\lambda k_1 + (1-\lambda)k_2, a) > \lambda f(k_1,a) + (1-\lambda)f(k_2,a).$$

By implication,

$$f(\lambda k_1 + (1-\lambda)k_2, a) - (\lambda k_1' + (1-\lambda)k_2') + (1-\delta)(\lambda k_1 + (1-\lambda)k_2)$$
$$= f(\lambda k_1 + (1-\lambda)k_2, a) + \lambda((1-\delta)k_1 - k_1') + (1-\lambda)((1-\delta)k_2 - k_2')$$
$$> \lambda\left[f(k_1,a) - k_1' + (1-\delta)k_1\right] + (1-\lambda)\left[f(k_2,a) - k_2' + (1-\delta)k_2\right].$$

Finally, by the strict concavity of $u(\cdot)$,

$$U(\lambda(k_1,k_1') + (1-\lambda)(k_2,k_2'), a)$$

$$\geq \lambda u\left[f(k_1,a)-k_1'+(1-\delta)k_1\right]+(1-\lambda)\left[f(k_2,a)-k_2'+(1-\delta)k_2\right]$$

$$=\lambda U(k_1,k_1',a)+(1-\lambda)U(k_2,k_2',a).$$

To show that $U(k,\cdot,a)$ is strictly concave on $\Gamma(k,a)$ for any $(k,a)\in X\times Z$, note simply that, for any distinct $k_1',k_2'\in\Gamma(k,a)$ and $\lambda\in[0,1]$,

$$\begin{aligned}U(k,\lambda k_1'+(1-\lambda)k_2',a)&=u\left[f(k,a)+(1-\delta)k-(\lambda k_1'+(1-\lambda)k_2')\right]\\&=u\left[\lambda(f(k,a)+(1-\delta)k-k_1')+(1-\lambda)(f(k,a)+(1-\delta)k-k_2')\right]\\&>\lambda u\left[f(k,a)+(1-\delta)k-k_1'\right]+(1-\lambda)u\left[f(k,a)+(1-\delta)k-k_2'\right]\\&=\lambda U(k,k_1',a)+(1-\lambda)U(k,k_2',a).\end{aligned}$$

B7 : **Differentiability Assumptions**

For any $a\in Z$, $U(\cdot,\cdot,a)$ is twice continuously differentiable on the interior of $\mathrm{Gr}(\Gamma;a)$ by the twice continuous differentiability of $u(\cdot)$ and $f(\cdot)$.

Preliminary Model Results

Now it follows from the results we proved in the previous chapter that:

1) **Existence and Uniqueness of Solution to Functional Equation**
   There exists a unique continuous and bounded solution V^* to the functional equation:

$$V^*(k,a) = \sup_{k' \in \Gamma(k,a)} \left[u[f(k,a) + (1-\delta)k - k'] + \beta \cdot \int_Z V^*(k',a') P(a,da') \right]$$

   for any $(k,a) \in X \times Z$. Furthermore, $V^*(\cdot,a)$ is strictly concave on X for any $a \in Z$.

2) **Continuous and Single-valued Policy Correspondence**
   There exists a continuous capital policy function $k^* : X \times Z \to X$ such that

$$V^*(k,a) = u[f(k,a) + (1-\delta)k - k^*(k,a)] + \beta \cdot \mathbb{E}\left[V^*(k^*(k,a),a') \mid a \right]$$

   for any $(k,a) \in X \times Z$. Similarly, there exists a continuous optimal consumption policy function $c^* : X \times Z \to \mathbb{R}_+$ defined as

$$c^*(k,a) = f(k,a) + (1-\delta)k - k^*(k,a)$$

   for any $(k,a) \in X \times Z$.

3) **Envelope Theorem for Value Function**
   For any $a \in Z$ and $k \in (0, \hat{k})$ such that $g(k,a) \in (0, f(k,a) + (1-\delta)k)$, we have

$$\nabla_k V^*(k,a) = \frac{\partial U(k, k^*(k,a), a)}{\partial k} = u'(c^*(k,a)) \cdot [f_k(k,a) + (1-\delta)] .$$

5.1.3 The Euler Equation

Under our model assumptions, we can show first that consumption is always positive. This in turn allows us to apply the envelope theorem to establish the Euler equation.

Lemma (Positivity of Consumption)
Consider the stochastic neoclassical growth model as given above. Then, the following hold true:

i) For any $a \in Z$, $c^*(0,a) = k^*(0,a) = 0$ and $c^*(k,a) > 0$ if $k \in (0,\hat{k}]$.

ii) For any $(k,a) \in (0,\hat{k}) \times Z$,

$$\nabla_k V^*(k,a) = u'\left[c^*(k,a)\right]\left(f_k(k,a) + (1-\delta)\right).$$

iii) For any $(k,a) \in X \times Z$ such that $k^*(k,a) > 0$,

$$\nabla_k V^*(k^*(k,a),a) = u'\left[c^*(k^*(k,a),a)\right]\left(f_k(k^*(k,a),a) + (1-\delta)\right).$$

Proof) For any $a \in Z$,

$$\Gamma(0,a) = [0, f(0,a) + (1-\delta)\cdot 0] = \{0\},$$

so we immediately have $k^*(0,a) = 0$ and $c^*(0,a) = 0$.

Now we show that consumption is always positive given a positive capital stock. Fix any $a \in Z$, and suppose that $c^*(k,a) = 0$ for some $k \in (0,\hat{k}]$. Then, by the flow budget constraint,

$$k^*(k,a) = f(k,a) + (1-\delta)k > 0.$$

Since $V^*(\cdot)$ is a continuous function on the compact set $X \times Z$, by the extreme value theorem there exists some $M > 0$ such that $|V^*| \leq M$ on $X \times Z$. This makes $\{V^*(\cdot,a')\}_{a' \in Z}$ a collection of concave functions that are uniformly bounded on

X. Since uniformly bounded collections of concave functions are uniformly equi-Lipschitz on any compact subset of their common domain, and X is a compact set, there exists some $L > 0$ such that

$$\left|V^*(k_1, a') - V^*(k_2, a')\right| \leq L|k_1 - k_2|$$

for any $k_1, k_2 \in X$ and $a' \in Z$.

Recall that we assumed $u'(c) \to +\infty$ as $c \downarrow 0$. By implication, there exists some $\eta \in (0, f(k,a) + (1-\delta)k)$ such that

$$u'(c) > L \quad \text{for any } c \leq \eta.$$

From the mean value theorem, we can see that there exists some $\tilde{\eta} \in (0, \eta]$ such that

$$-u(\eta) = u(0) - u(\eta) = -u'(\tilde{\eta})\eta.$$

Since $\tilde{\eta} \leq \eta$, it follows that

$$-u(\eta) < -L\eta.$$

Now note that

$$\begin{aligned} V^*(k,a) - \left[u'(\eta) + \beta \int_Z V^*(k^*(k,a) - \eta, a')P(a, da')\right] \\ = -u(\eta) + \beta \int_Z (V^*(k^*(k,a), a') - V^*(k^*(k,a) - \eta, a'))P(a, da') \\ \leq -u(\eta) + \beta L\eta < L\eta(\beta - 1) < 0, \end{aligned}$$

or in other words,

$$V^*(k,a) < u'(\eta) + \beta \int_Z V^*(k^*(k,a) - \eta, a')P(a, da').$$

Since $k^*(k,a) - \eta \in \Gamma(k,a)$ by construction, we have

$$V^*(k,a) < u'(\eta) + \beta \int_Z V^*(k^*(k,a) - \eta, a')P(a, da') \leq V^*(k,a),$$

a contradiction. Therefore, $c^*(k,a) > 0$ if $k > 0$.

Now choose any $(k,a) \in (0,\hat{k}) \times Z$. Then, because $c^*(k,a) > 0$,

$$k^*(k,a) = f(k,a) + (1-\delta)k - c^*(k,a) < f(k,a) + (1-\delta)k.$$

This shows us that $k^*(k,a) \in [0, f(k,a) + (1-\delta)k)$. Define

$$\epsilon = f(k,a) + (1-\delta)k - k^*(k,a) > 0.$$

By the continuity of the mapping $k' \mapsto f(k',a) + (1-\delta)k'$ and the fact that k is in the interior of $(0,\bar{k})$, there exists some $\eta > 0$ such that

$$\left| f(k',a) + (1-\delta)k' - \left(f(k,a) + (1-\delta)k\right) \right| < \epsilon$$

for any $k' \in B(k,\eta)$. It follows that, for any $k' \in B(k,\eta)$,

$$f(k',a) + (1-\delta)k' > f(k,a) + (1-\delta)k - \epsilon = k^*(k,a)$$

and therefore

$$k^*(k,a) \in \Gamma(k',a).$$

We have found a neighborhood $B(k,\eta)$ of k such that $k^*(k,a) \in \Gamma(k',a)$ for any $k' \in B(k,\eta)$; therefore, in light of the theorem establishing the differentiability of the value function, $V^*(\cdot,a)$ is differentiable at k with derivative given as

$$\boldsymbol{\nabla}_k V^*(k,a) = u'\left[c^*(k,a)\right]\left(f_k(k,a) + 1 - \delta\right).$$

Finally, for any $(k,a) \in X \times Z$ such that $k^*(k,a) > 0$, since $k^*(k,a) \in (0,\bar{k})$ in this case, we have

$$\boldsymbol{\nabla}_k V^*(k^*(k,a),a') = u'\left[c^*(k^*(k,a),a')\right]\left(f_k(k^*(k,a),a') + 1 - \delta\right)$$

for any $a' \in Z$.

Q.E.D.

We can now establish the Euler equation. Choose any $(k,a) \in X \times Z$ such that $k^*(k,a) > 0$. Then, by the first order conditions for maximization,

$$-u'\left[c^*(k,a)\right] + \beta \cdot \int_Z \nabla_k V^*(k^*(k,a),a')P(a,da') = 0,$$

where we can interchange the order of integration and differentiation by appealing to the bounded convergence theorem. We saw above that

$$\nabla_k V^*(k^*(k,a),a') = u'\left[c^*(k^*(k,a),a')\right]\left(f_k(k^*(k,a),a') + 1 - \delta\right)$$

for any $a' \in Z$, so that

$$u'\left[c^*(k,a)\right] = \beta \int_Z u'\left[c^*(k^*(k,a),a')\right]\left(f_k(k^*(k,a),a') + 1 - \delta\right)P(a,da').$$

This is the Euler equation.

5.1.4 Monotonicity of Policy Functions

Here we show that the optimal policy functions in capital and productivity are strictly increasing in their arguments (the latter for some special specifications).

Theorem (Monotonicity of Consumption and Capital Policy Functions) Consider the stochastic neoclassical growth model as given above, and assume that $k^*(k,a) > 0$ for any $(k,a) \in (0,\hat{k}] \times Z$. Then, the following results hold true:

i) For any $a \in Z$, $k^*(\cdot,a)$ and $c^*(\cdot,a)$ are strictly increasing on X.

ii) Suppose that the productivity shocks are drawn independently from the distribution λ every period, so that $P(a,\cdot) = \lambda$ for any $a \in Z$. Then, for any $k \in (0,\hat{k}]$, $k^*(k,\cdot)$ and $c^*(k,\cdot)$ are strictly increasing on Z.

Proof) i) Fix $a \in Z$ and choose any $k_1, k_2 \in X$ such that $k_1 < k_2$. We consider two cases. First, suppose $0 < k_1 < k_2$. Because $k^*(k_2, a), k^*(k_1, a) > 0$ by assumption, we can see that the Euler equation holds:

$$u'[c^*(k_1, a)] = \beta \int_Z \nabla_k V^*(k^*(k_1, a), a') P(a, da')$$

$$u'[c^*(k_2, a)] = \beta \int_Z \nabla_k V^*(k^*(k_2, a), a') P(a, da').$$

Suppose $c^*(k_1, a) \geq c^*(k_2, a)$. Then,

$$\beta \int_Z \nabla_k V^*(k^*(k_1, a), a') P(a, da') = u'[c^*(k_1, a)]$$

$$\leq u'[c^*(k_2, a)] = \beta \int_Z \nabla_k V^*(k^*(k_2, a), a') P(a, da'),$$

and as such

$$\int_Z \left[\nabla_k V^*(k^*(k_2, a), a') - \nabla_k V^*(k^*(k_1, a), a') \right] P(a, da') \geq 0.$$

$V^*(\cdot, a')$ is strictly concave for any $a' \in Z$, so $\nabla_k V^*(\cdot, a')$ is strictly decreasing on $(0, \hat{k})$. It follows that

$$\int_Z \left[\nabla_k V^*(k^*(k_2, a), a') - \nabla_k V^*(k^*(k_1, a), a') \right] P(a, da') \leq 0.$$

If this inequality holds as an equality, then by the vanishing property of non-negative functions, we would have $\nabla_k V^*(k^*(k_2, a), a') = \nabla_k V^*(k^*(k_1, a), a')$ for $P(a, \cdot)$-a.e. $a' \in Z$, which contradicts the strictly decreasing nature of each $\nabla_k V^*(\cdot, a')$. Therefore,

$$\int_Z \left[\nabla_k V^*(k^*(k_2, a), a') - \nabla_k V^*(k^*(k_1, a), a') \right] P(a, da') < 0$$

$$\leq \int_Z \left[\nabla_k V^*(k^*(k_2, a), a') - \nabla_k V^*(k^*(k_1, a), a') \right] P(a, da'),$$

which is another contradiction. It follows that $c^*(k_1, a) < c^*(k_2, a)$. Finally, for any $k \in (0, \hat{k}]$, we have

$$c^*(0, a) = 0 < c^*(k, a).$$

This proves that $c^*(\cdot, a)$ is strictly increasing on X.

Moving onto the capital policy function, choose any $k_1, k_2 \in X$ such that $0 < k_1 < k_2$. Then, we know from the Euler equation and the monotonicity of the consumption policy function in capital stock that

$$\beta \int_Z \nabla_k V^*(k^*(k_1, a), a') P(a, da') = u'\left[c^*(k_1, a)\right]$$
$$> u'\left[c^*(k_2, a)\right] = \beta \int_Z \nabla_k V^*(k^*(k_2, a), a') P(a, da'),$$

so that

$$\int_Z \left(\nabla_k V^*(k^*(k_1, a), a') - \nabla_k V^*(k^*(k_2, a), a')\right) P(a, da') > 0.$$

If $k^*(k_1, a) \geq k^*(k_2, a)$, then by the strict concavity of $V^*(\cdot, a')$,

$$\nabla_k V^*(k^*(k_1, a), a') - \nabla_k V^*(k^*(k_2, a), a') \leq 0$$

for any $a' \in Z$. This implies

$$\int_Z \left(\nabla_k V^*(k^*(k_1, a), a') - \nabla_k V^*(k^*(k_2, a), a')\right) P(a, da') \leq 0,$$

a contradiction. Therefore, we must have $k^*(k_1, a) < k^*(k_2, a)$. Finally, for any $k \in (0, \hat{k}]$,

$$k^*(0, a) = 0 < k^*(k, a).$$

This proves that $k^*(\cdot, a)$ is strictly increasing on X.

ii) Now suppose that the productivity shocks are drawn independently each period from the distribution λ. Fix $k \in (0, \hat{k}]$, and choose any $a_1, a_2 \in Z$ such that $a_1 < a_2$. Suppose $k^*(k, a_1) \geq k^*(k, a_2)$. Then,

$$c^*(k, a_1) = f(k, a_1) + (1 - \delta)k - k^*(k, a_1) < f(k, a_2) + (1 - \delta)k - k^*(k, a_2) = c^*(k, a_2).$$

Since $k^*(k, a_2) > 0$ by assumption, the Euler equation now tells us that

$$\beta \int_Z \nabla_a V^*(k^*(k, a_1), a') d\lambda(a') = u'[c^*(k, a_1)]$$

$$> u'[c^*(k, a_2)] = \beta \int_Z \nabla_a V^*(k^*(k, a_2), a') d\lambda(a').$$

By the strict concavity of $V^*(\cdot, a')$,

$$\nabla_a V^*(k^*(k, a_1), a') \leq \nabla_a V^*(k^*(k, a_2), a')$$

for any $a' \in Z$, so that

$$\int_Z \nabla_a V^*(k^*(k, a_1), a') d\lambda(a') \leq \int_Z \nabla_a V^*(k^*(k, a_2), a') d\lambda(a').$$

It follows that

$$\beta \int_Z \nabla_a V^*(k^*(k, a_1), a') d\lambda(a') < \beta \int_Z \nabla_a V^*(k^*(k, a_1), a') d\lambda(a'),$$

a contradiction. Therefore, $k^*(k, a_1) < k^*(k, a_2)$, and $k^*(k, \cdot)$ is strictly increasing on Z.

We can show that the consumption policy function is strictly increasing in TFP in a similar manner. Choose any $a_1, a_2 \in Z$ such that $a_1 < a_2$. Since $k^*(k, a_1) < k^*(k, a_2)$, we have

$$\int_Z \nabla_a V^*(k^*(k, a_1), a') d\lambda(a') > \int_Z \nabla_a V^*(k^*(k, a_2), a') d\lambda(a'),$$

and the Euler equation implies that

$$u'[c^*(k, a_1)] > u'[c^*(k, a_2)].$$

Therefore, $c^*(k, a_1) < c^*(k, a_2)$, which proves that $c^*(k, \cdot)$ is strictly increasing on Z.

Q.E.D.

5.1.5 Stationary Distribution of Capital Stock

We want to study some convergence properties of the optimal capital path $\{K_t\}_{t\in\mathbb{N}}$. To start, note that $\{(K_t, A_t)\}_{t\in\mathbb{N}}$ is a Markov process on $X \times Z$; heuristically, this is because K_{t+1} is determined by K_t and A_t via the policy function g, while the distribution of A_{t+1} given all the information in the stochastic process up to time t depends only on A_t.

Formally, define the function $Q : (X \times Z) \times (\mathcal{X} \bigotimes \mathcal{Z}) \to [0, 1]$ as

$$Q((k, a), B) = \int_{X\times Z} I_B(k^*(k, a), a')P(a, da')$$

for any $(k, a) \in X \times Z$ and $B \in \mathcal{X} \bigotimes \mathcal{Z}$. We know that this is a well-defined Markov kernel on $(X \times Z, \mathcal{X} \bigotimes \mathcal{Z})$, and since $g(\cdot)$ is a continuous function and P possesses the Feller property, we can also immediately conclude that Q also has the Feller property.

If, in addition, productivity shocks are drawn independently from the same distribution λ on $(Z, \mathcal{Z})$, then $k^*(\cdot)$ is monotonic and therefore Q is a monotonic kernel. To prove that Q converges weakly to a unique stationary distribution, it thus remains to verify the stable mixing condition. To this end, we make the following assumption:

> We assume that the shocks are drawn independently from a distribution v on $(Z, \mathcal{Z})$ such that
>
> $$v(\{\underline{a}\}),\ v(\{\overline{a}\}) > 0.$$

In other words, the lowest or highest productivity shocks can be drawn with positive probability at any time. This is a stronger assumption than necessary to facilitate the derivation of the stationary distribution; it may be relaxed to the assumption that v has full support on Z. We can now prove the existence of a unique stationary distribution in a couple of steps:

Lemma (Existence of Lower Bound for Fixed Points)
Consider the stochastic neoclassical growth model as given above, and assume that $k^*(k,a) > 0$ for any $(k,a) \in (0,\hat{k}] \times Z$. For any $a \in Z$ such that $v(\{a\}) > 0$, there exists some $\epsilon > 0$ such that $k^*(k,a) > k$ for any $k \in (0,\epsilon]$[1].

Proof) First, let $k \in (0,\hat{k}]$ be some fixed point of $k^*(\cdot,a)$, so that $k^*(k,a) = k > 0$. Then, by the Euler equation,

$$u'\left[c^*(k,a)\right] = \beta \int_Z u'\left[c^*(k,a')\right]\left(f_k(k,a') + 1 - \delta\right) dv(a')$$

$$\geq \beta u'\left[c^*(k,a)\right]\left(f_k(k,a) + 1 - \delta\right) v(\{a\}).$$

By implication,

$$1 \geq \beta v(\{a\})\left(f_k(k,a) + 1 - \delta\right) > 0.$$

Now suppose that we can construct a sequence $\{k_n\}_{n \in N_+}$ of fixed points of $k^*(\cdot,a)$ that decreases to 0. For any $n \in N_+$,

$$1 \geq \beta v(\{a\})\left(f_k(k_n,a) + 1 - \delta\right) > 0,$$

but by the Inada conditions,

$$\beta v(\{a\})\left(f_k(k_n,a) + 1 - \delta\right) > 1$$

for large enough n. This is a contradiction, so there must exist some $\epsilon > 0$ such that either $k^*(k,a) < k$ for any $k \in (0,\epsilon]$ or $k^*(k,a) > k$ for any $k \in (0,\epsilon]$.

Suppose, for the sake of contradiction, that $k^*(k,a) < k$ for any $k \in (0,\epsilon]$. Then, choosing any initial value $k_0 \in (0,\epsilon]$, we can construct the sequence $\{k_t\}_{t \in \mathbb{N}}$ as

$$k_{t+1} = k^*(k_t,a)$$

for any $t \geq 0$. $0 < k_1 = k^*(k_0,a) < k_0$ because $k_0 > 0$. Now suppose $0 < k_t <$

[1]This is the content of lemmas 3.1 and 3.2 in Brock and Mirman (1972).

$k_{t-1} \leq \epsilon$ for some $t \geq 1$. Since $k_t \in (0, \epsilon]$, it follows that

$$0 < k_{t+1} = k^*(k_t, a) < k_t < \epsilon,$$

so by induction, $\{k_t\}_{t \in \mathbb{N}}$ is a strictly decreasing sequence contained in $(0, \epsilon]$. This indicates that $\{k_t\}_{t \in \mathbb{N}}$ is a convergent sequence with limit $L \in [0, \epsilon]$. Taking $t \to \infty$ on both sides of the equation

$$k_{t+1} = k^*(k_t, a)$$

and using the continuity of $k^*(\cdot, a)$ shows us that $k^*(L, a) = L$. This means that $L \notin (0, \epsilon]$, so that $L = 0$; in other words, $k_t \searrow 0$ as $t \to \infty$.

By the Euler equation,

$$\begin{aligned} u'[c^*(k_t, a)] &= \beta \int_Z u' \left[c^*(k_{t+1}, a')\right] \left(f_k(k_{t+1}, a') + 1 - \delta\right) dv(a') \\ &\geq \beta v(\{a\}) u'[c^*(k_{t+1}, a)] (f_k(k_{t+1}, a) + 1 - \delta) \\ &> \beta v(\{a\}) u'[c^*(k_t, a)] (f_k(k_{t+1}, a) + 1 - \delta) \end{aligned}$$

for any $t \geq 0$, where the last inequality follows because $k_t > k_{t+1}$ implies $c^*(k_t, a) > c^*(k_{t+1}, a)$. Therefore,

$$1 > \beta v(\{a\}) (f_k(k_{t+1}, a) + 1 - \delta) > 0.$$

for any $t \geq 0$. However, the Inada conditions for the production function, as well as the fact that $k_t \searrow 0$ as $t \to \infty$, indicates that

$$\beta v(\{a\}) (f_k(k_{t+1}, a) + 1 - \delta) > 1$$

for large enough t, a contradiction. Therefore, it must be the case that $k^*(k, a) > k$ for any $k \in (0, \epsilon]$, which concludes our proof.

Q.E.D.

Lemma (Properties of Fixed Points)
Consider the stochastic neoclassical growth model as given above, and assume that $k^*(k,a) > 0$ for any $(k,a) \in (0,\hat{k}] \times Z$. Defining the functions $\underline{h}, \overline{h} : [0,\hat{k}] \to \mathbb{R}$ as

$$\underline{h}(k) = k^*(k,\underline{a}), \quad \overline{h}(k) = k^*(k,\overline{a}).$$

Then, letting $\underline{H}$ and $\overline{H}$ be the set of all interior fixed points

$$\underline{H} = \{k \in (0,\hat{k}) \mid \underline{h}(k) = k\}, \quad \overline{H} = \{k \in (0,\hat{k}) \mid \overline{h}(k) = k\},$$

$\underline{H}$ and $\overline{H}$ are non-empty compact sets such that $\underline{k} < \overline{k}$ for any $\underline{k} \in \underline{H}$ and $\overline{k} \in \overline{H}$[2].

Proof) **Existence of Interior Fixed Points**

Note that $\overline{h}(\cdot)$ is a continuous function on $[0,\hat{k}]$ such that

$$0 < k^*(k,\overline{a}) < f(k,\overline{a}) + (1-\delta)k \leq \hat{k}$$

whenever $k \in (0,\hat{k}]$, the first inequality holding by assumption and the second by the postivity of consumption. We can see immediately that $k^*(\hat{k},\overline{a}) < \hat{k}$. Since $v(\{\overline{a}\}) > 0$, the preceding result tells us that there exists some $\overline{\epsilon} > 0$ such that $\overline{h}(k) > k$ for any $k \in (0,\overline{\epsilon}]$. This implies that

$$\overline{h}(\hat{k}) - \hat{k} < 0 < \overline{h}(\overline{\epsilon}) - \overline{\epsilon}.$$

The mapping $k \mapsto \overline{h}(k) - k$ is a continuous function, so by the intermediate value theorem, there exists some $\overline{k} \in (\overline{\epsilon},\hat{k})$ such that

$$\overline{h}(\overline{k}) = \overline{k}.$$

By definition, $\overline{k} \in \overline{H}$, so that $\overline{H}$ is a non-empty set. Furthermore, we can see that $\overline{H}$ is bounded below by $\overline{\epsilon}$ and above by $\hat{k}$.

An identical argument reveals that there exists some $\underline{\epsilon} > 0$ such that $k^*(k,\underline{a}) > k$ for any $k \in (0,\underline{\epsilon}]$, and that $\underline{H}$ is non-empty and bounded below by $\underline{\epsilon}$.

[2]This is the content of lemma 3.4 in Brock and Mirman (1972).

Compactness of Set of Fixed Points

We now show that $\overline{H}$ is compact via sequential compactness. Let $\{k_n\}_{n\in N_+}$ be a sequence contained in $\overline{H}$. Since $[\overline{\epsilon}, \hat{k}]$ is bounded, the sequence $\{k_n\}_{n\in N_+}$ is bounded. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\{k_n\}_{n\in N_+}$; for simplicity, let $\{k_n\}_{n\in N_+}$ itself be this subsequence, and let its limit be $\overline{k}$.

Since $k_n \in \overline{H}$, we have

$$\overline{h}(k_n) = k_n \quad \text{for all } n.$$

By continuity of $\overline{h}$, taking limits yields

$$\overline{h}(\overline{k}) = \overline{k}.$$

Finally, we can see that $\overline{k} \in [\overline{\epsilon}, \hat{k}) \subset (0, \hat{k})$, where $\overline{k} \neq \hat{k}$ because we know that $\hat{k}$ is not a fixed point of $\overline{h}(\cdot)$. Thus, $\overline{k} \in \overline{H}$, and $\overline{H}$ is sequentially compact and hence compact.

Since $\overline{H}$ is compact, $\inf \overline{H}$ and $\sup \overline{H}$ exist and are elements of $\overline{H}$. An identical argument applies to $\underline{H}$.

Ordering of Fixed Points Across $\underline{H}$ and $\overline{H}$

Choose any $\underline{k} \in \underline{H}$ and $\overline{k} \in \overline{H}$. By definition, $k^*(\overline{k}, \overline{a}) = \overline{k} > 0$, so by the Euler equation, we have

$$\begin{aligned} u'\left[c^*(\overline{k}, \overline{a})\right] &= \beta \int_Z u'\left[c^*(\overline{k}, a')\right]\left(f_k(\overline{k}, a') + 1 - \delta\right) dv(a') \\ &\geq \beta u'\left[c^*(\overline{k}, \overline{a})\right] \int_Z \left(f_k(\overline{k}, a') + 1 - \delta\right) dv(a'), \end{aligned}$$

where we used the fact that $c^*(\overline{k}, a') \leq c^*(\overline{k}, \overline{a})$ for any $a' \in Z$ ($c^*(\overline{k}, \cdot)$ is strictly increasing on Z because $\overline{k} > 0$). If this inequality holds as an equality, then the vanishing property of non-negative functions implies that

$$\left(u'\left[c^*(\overline{k}, a')\right] - u'\left[c^*(\overline{k}, \overline{a})\right]\right)\left(f_k(\overline{k}, a') + 1 - \delta\right) = 0$$

for v-a.e. $a' \in Z$, which contradicts the fact that $u'\left[c^*(\overline{k},\underline{a})\right] > u'\left[c^*(\overline{k},\overline{a})\right]$ and $v(\{\underline{a}\}) > 0$. Thus, the inequality must hold strictly, and we have

$$1 > \beta \int_Z \left(f_k(\overline{k},a') + 1 - \delta\right) dv(a').$$

An identical line of reasoning leads us to the inequality

$$1 < \beta \int_Z \left(f_k(\underline{k},a') + 1 - \delta\right) dv(a').$$

Putting the two inequalities together yields

$$\int_Z \left(f_k(\underline{k},a') + 1 - \delta\right) dv(a') > \int_Z \left(f_k(\overline{k},a') + 1 - \delta\right) dv(a').$$

If $\underline{k} \geq \overline{k}$, then

$$f_k(\underline{k},a') + 1 - \delta \leq f_k(\overline{k},a') + 1 - \delta$$

for any $a' \in Z$ and therefore

$$\int_Z \left(f_k(\underline{k},a') + 1 - \delta\right) dv(a') \leq \int_Z \left(f_k(\overline{k},a') + 1 - \delta\right) dv(a'),$$

which contradicts the strict inequality above. As such, we must have $\underline{k} < \overline{k}$.

Q.E.D.

We now define

$$\underline{k}_M = \sup \underline{H}, \quad \underline{k}_m = \inf \underline{H},$$

$$\overline{k}_M = \sup \overline{H}, \quad \overline{k}_m = \inf \overline{H}.$$

Because fixed points for $\overline{h}(\cdot)$ are strictly larger than any fixed point of $\underline{h}(\cdot)$, we can see that $\underline{k}_m \leq \underline{k}_M < \overline{k}_m \leq \overline{k}_M$, that is, the largest fixed point when we continue to face the lowest productivity shock is strictly smaller than the smallest fixed point when we continue to face the highest productivity shock.

The proof of the stable mixing condition is now almost complete. Define 2-cell $[c,d]$

as

$$[c,d] = [\underline{k}_m, \overline{k}_M] \times Z.$$

We will show that $[c,d]$ is a stable set for the kernel Q, and that any Markov chain with kernel Q enters this stable set with probability 1 given non-trivial initial values.

Lemma (Obtaining a Stable Set)
Consider the stochastic neoclassical growth model as given above, and assume that $k^*(k,a) > 0$ for any $(k,a) \in (0, \hat{k}] \times Z$. Let $[c,d]$ be the 2-cell defined above. Then, $[c,d]$ is a stable set for Q, and for any $s_0 = (k_0, a_0) \in (0, \hat{k}] \times Z$,

$$\lim_{t \to \infty} Q^t(s_0, [c,d]) = 1.$$

Proof) We first show that $[c,d]$ is stable for Q. For any $(k,a) \in [c,d]$, by the monotonicity of the policy function $k^*(\cdot)$, we have

$$k^*(k,a) \leq k^*(\overline{k}_M, \overline{a}) = \overline{k}_M.$$

Similarly,

$$k^*(k,a) \geq k^*(\underline{k}_m, \underline{a}) = \underline{k}_m.$$

Therefore,

$$Q((k,a),[c,d]) = I_{k^*(k,a) \in [\underline{k}_m, \overline{k}_M]} v(Z) = 1.$$

Now choose any $s_0 = (k_0, a_0) \in (0, \hat{k}] \times Z$. Let the sequence $\{K_t, A_t\}_{t \in \mathbb{N}}$ of random variables be initialized as $K_0 = k_0, A_0 = a_0$ and generated according to the recursion

$$K_{t+1} = k^*(K_t, A_t), \quad A_{t+1} \sim v$$

for any $t \geq 0$. It follows that $\{K_t, A_t\}_{t \in \mathbb{N}}$ is a Markov chain with kernel Q and

initial distribution δ_{s_0}; thus, the distribution of (K_t, A_t) is $Q^t(s_0, \cdot)$. To show that

$$\lim_{t\to\infty} Q^t(s_0, [c,d]) = 1,$$

it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}((K_t, A_t) \in [c,d]) = 1.$$

Since A_t takes values in Z and $[c,d] = [\underline{k}_m, \overline{k}_M] \times Z$, this reduces to showing that

$$\lim_{t\to\infty} \mathbb{P}\left(K_t \in [\underline{k}_m, \overline{k}_M]\right) = 1.$$

Define the set

$$\overline{\Omega} = \bigcap_{T\in\mathbb{N}} \bigcup_{t\geq T} \{A_t = \overline{a}\}.$$

Then, $\overline{\Omega}$ has probability 1 because $v(\{\overline{a}\}) > 0$ implies that

$$\begin{aligned}
\mathbb{P}\left(\overline{\Omega}^c\right) &= \mathbb{P}\left(\bigcup_{T\in\mathbb{N}} \bigcap_{t\geq T} \{A_t < \overline{a}\}\right) \\
&\leq \sum_{T=0}^{\infty} \mathbb{P}\left(\bigcap_{t\geq T} \{A_t < \overline{a}\}\right) \\
&= \sum_{T=0}^{\infty} \lim_{\tau\to\infty} \prod_{t=T}^{T+\tau} \mathbb{P}(A_t < \overline{a}) \\
&= \sum_{T=0}^{\infty} \lim_{\tau\to\infty} (1 - v(\{\overline{a}\}))^{\tau+1} = 0,
\end{aligned}$$

where the first inequality follows from countable subadditivity and the second equality follows from sequential continuity. Similarly,

$$\underline{\Omega} = \bigcap_{T\in\mathbb{N}} \bigcup_{t\geq T} \{A_t = \underline{a}\}$$

has probability 1 because $v(\{\underline{a}\}) > 0$. It follows that

$$\Omega_0 = \overline{\Omega} \cap \underline{\Omega}$$

is an almost sure event.

Choose any $\omega \in \Omega_0$, and consider the realization $\{k_t, a_t\}_{t \in \mathbb{N}}$ defined as $k_t = K_t(\omega)$, $a_t = A_t(\omega)$ for any $t \in \mathbb{N}$. We consider three cases:

Case 1 $k_0 \in [\underline{k}_m, \overline{k}_M]$

In this case, $k_t \in [\underline{k}_m, \overline{k}_M]$ for any $t \in \mathbb{N}$ by the stability of $[c, d]$, so that

$$I_{[\underline{k}_m, \overline{k}_M]}(k_t) = 1 \quad \text{for any } t \in \mathbb{N}.$$

Case 2 $0 < k_0 < \underline{k}_m$

In this case, we already know that $k_0 < \overline{k}_M$; if $k_t < \overline{k}_M$ for some $t \geq 0$, then

$$k_{t+1} = k^*(k_t, a_t) \leq k^*(k_t, \overline{a}) < k^*(\overline{k}_M, \overline{a}) = \overline{k}_M,$$

so that $\{k_t\}_{t \in \mathbb{N}}$ is contained in $(0, \overline{k}_M]$.

Now suppose $k_t < \underline{k}_m$ for any $t \in \mathbb{N}$. Then, because $k^*(k, \underline{a}) > k$ for any $0 < k < \underline{k}_m$ (otherwise, there is another positive fixed point of $k^*(\cdot, \underline{a})$ smaller than $\underline{k}_m$, which contradicts the minimality of $\underline{k}_m$), it follows that

$$k_{t+1} = k^*(k_t, a_t) \geq k^*(k_t, \underline{a}) > k_t$$

for any $t \in \mathbb{N}$. This shows us that $\{k_t\}_{t \in \mathbb{N}}$ is an increasing sequence bounded above by $\underline{k}_m$, so that it converges to some limit $L \leq \underline{k}_m$. Taking $t \to \infty$ in the inequality

$$k_{t+1} \geq k^*(k_t, \underline{a}) > k_t$$

shows us that $k^(L, \underline{a}) = L$, and as such that $L = \underline{k}_m$.

The mapping

$$k \mapsto k^*(k, \overline{a}) - \underline{k}_m$$

is a continuous function, and

$$k^*(\underline{k}_m, \overline{a}) - \underline{k}_m = k^*(\underline{k}_m, \overline{a}) - k^*(\underline{k}_m, \underline{a}) > 0,$$

so there exists some $\epsilon > 0$ such that

$$|k^*(k, \overline{a}) - \underline{k}_m - (k^*(\underline{k}_m, \overline{a}) - \underline{k}_m)| < k^*(\underline{k}_m, \overline{a}) - \underline{k}_m$$

and in particular

$$k^*(k, \overline{a}) > \underline{k}_m,$$

for any $k \in B(\underline{k}_m, \epsilon)$. Because $k_t \nearrow \underline{k}_m$ as $t \to \infty$, there exists some $T \in N_+$ such that

$$\underline{k}_m - \epsilon < k_t \leq \underline{k}_m$$

for any $t \geq T$. Furthermore, by the way we chose ω, there exists some $t \geq T$ such that $a_t = \overline{a}$, so that

$$k_{t+1} = k^*(k_t, a_t) = k^*(k_t, \overline{a}) > \underline{k}_m$$

because $k_t \in B(\underline{k}_m, \epsilon)$. This contradicts our initial supposition that $k_t < \underline{k}_m$ for any $t \in \mathbb{N}$, so it must be the case that $k_T \geq \underline{k}_m$ for some $T \in N_+$. Because $[\underline{k}_m, \overline{k}_M] \times Z$ is a stable set for Q, it follows that

$$I_{[\underline{k}_m, \overline{k}_M]}(k_t) = 1 \quad \text{for any } t \geq T.$$

Case 3 $\overline{k}_M < k_0 \leq \hat{k}$

We can immediately see that $\underline{k}_m < k_0$; if $k_t > \underline{k}_m$ for some $t \geq 0$, then

$$k_{t+1} = k^*(k_t, a_t) \geq k^*(k_t, \underline{a}) > k^*(\underline{k}_m, \underline{a}) = \underline{k}_m,$$

so that, by induction, $\{k_t\}_{t\in\mathbb{N}}$ is contained in $[\underline{k}_m, \hat{k}]$.

Suppose that $\overline{k}_M < k_t$ for any $t \in N_+$. In this case, because $k^*(k, \overline{a}) < k$ for any $k > \overline{k}_M$ (otherwise, there exists a fixed point of $k^*(\cdot, \overline{a})$ greater than $\overline{k}_M$, contradicting the maximality of $\overline{k}_M$), we can see that

$$k_{t+1} = k^*(k_t, a_t) \leq k^*(k_t, \overline{a}) < k_t$$

for any $t \in \mathbb{N}$, making $\{k_t\}_{t\in\mathbb{N}}$ a decreasing sequence bounded below by $\overline{k}_M$. This indicates that it converges toward some limit $L \geq \overline{k}_M$, and taking $t \to \infty$ on the chain of inequalities above shows us that $k^*(L, \overline{a}) = L$. As such, $k_t \searrow L = \overline{k}_M$.

The mapping

$$k \mapsto \overline{k}_M - k^*(k, \underline{a})$$

is a continuous function, and

$$\overline{k}_M - k^*(\overline{k}_M, \underline{a}) > 0,$$

so there exists some $\epsilon > 0$ such that

$$\overline{k}_M - k^*(k, \underline{a}) > 0$$

for any $k \in B(\overline{k}_M, \epsilon)$. Because $k_t \searrow \overline{k}_M$ as $t \to \infty$, there exists some $T \in N_+$ such that

$$\overline{k}_M + \epsilon > k_t \geq \overline{k}_M$$

for any $t \geq T$. Furthermore, by the way we chose ω, there exists some $t \geq T$ such that $a_t = \underline{a}$, so that

$$k_{t+1} = k^*(k_t, a_t) = k^*(k_t, \underline{a}) < \overline{k}_M$$

because $k_t \in B(\overline{k}_M, \epsilon)$. This contradicts our initial supposition that $k_t > \overline{k}_M$ for any $t \in \mathbb{N}$, so it must be the case that $k_T \leq \overline{k}_M$ for some $T \in N_+$. Because

$[\underline{k}_m, \overline{k}_M] \times Z$ is a stable set for Q, it follows that

$$I_{[\underline{k}_m, \overline{k}_M]}(k_t) = 1 \quad \text{for any } t \geq T.$$

In any case, we can find some $T \in N_+$ such that

$$I_{[\underline{k}_m, \overline{k}_M]}(k_t) = 1 \quad \text{for any } t \geq T,$$

so that

$$\lim_{t \to \infty} I_{[\underline{k}_m, \overline{k}_{M<}]}(k_t) = 1.$$

This holds for any $\omega \in \Omega_0$, so by the bounded convergence theorem,

$$\lim_{t \to \infty} \mathbb{P}\Big(K_t \in [\underline{k}_m, \overline{k}_M]\Big) = \mathbb{P}\left(\lim_{t \to \infty} I_{[\underline{k}_m, \overline{k}_M]} \circ K_t\right) = 1.$$

This in turn proves that

$$\lim_{t \to \infty} Q^t(s_0, [c, d]) = 1.$$

Q.E.D.

Now that we have shown that $[c, d]$ is a stable set for Q, we need to show that it satisfies the splitting condition. To this end, we define

$$\hat{s} = \left(\frac{\underline{k}_M + \overline{k}_m}{2},\, \underline{a}\right) \in (\underline{k}_M, \overline{k}_m) \times Z$$

$$\subset [\underline{k}_m, \overline{k}_M] \times Z = [c, d].$$

To intuitively see why the splitting condition holds with $\hat{s}$, suppose we start from the worst initial condition $(k_0, a_0) = (\underline{k}_m, \underline{a})$.If we continue to receive the highest productivity shocks $\overline{a}$ period over period, capital accumulation will follow the path generated by $k^*(\cdot, \overline{a})$, and eventually move very close to its smallest fixed point $\overline{k}_m$. This means that,

in finite time, capital stock will pass $\frac{\underline{k}_M+\overline{k}_m}{2}$, and thus we will reach the 2-cell $[\hat{s}, d]$ in finite time with positive probability.

Conversely, suppose we start from the best initial condition $(k_0, a_0) = (\overline{k}_M, \overline{a})$. In this case, if we continue to face the worst productivity shocks, capital stock gradually decreases until it reaches the neighborhood of $\underline{k}_M$; at this point, we will have reached the 2-cell $[c, \hat{s}]$ in finite time.

The formal argument is given below:

Lemma (The Splitting Condition for Capital)
Consider the stochastic neoclassical growth model as given above, and assume that $k*(k, a) > 0$ for any $(k, a) \in (0, \hat{k}] \times Z$. Then, there exists an $\epsilon > 0$ and $N \in N_+$ such that

$$Q^N(c, [\hat{s}, d]) \geq \epsilon, \quad Q^N(d, [c, \hat{s}]) \geq \epsilon.$$

Proof) We generate the process $\{k_t, a_t\}_{t\in\mathbb{N}}$ as follows. First we put $(k_0, a_0) = c = (\underline{k}_m, \underline{a})$ and, for any $t \in \mathbb{N}$, we define

$$k_{t+1} = k^*(k_t, a_t), \quad a_{t+1} = \overline{a}.$$

By construction, $k_1 = \underline{k}_m$ and

$$k_{t+1} = k^*(k_t, \overline{a}) \quad \text{for any } t \geq 1.$$

Note that $k_1 < \overline{k}_m$. Suppose $k_t < \overline{k}_m$ for some $t \geq 1$. Then,

$$k_{t+1} = k^*(k_t, \overline{a}) < k^*(\overline{k}_m, \overline{a}) = \overline{k}_m,$$

so that $\{k_t\}_{t\in\mathbb{N}}$ is contained in the interval $[\underline{k}_m, \overline{k}_m)$. Since $k^*(k, \overline{a}) > k$ for any $k < \overline{k}_m$, it follows that

$$k_{t+1} = k^*(k_t, \overline{a}) < k_t$$

and therefore that $\{k_t\}_{t\in\mathbb{N}}$ is a strictly increasing sequence with upper bound $\overline{k}_m$. This implies that there exists some limit $L \leq \overline{k}_m$ such that $k_t \nearrow L$ as $t \to \infty$; taking limits on the chain of inequalities above shows us that $k^*(L, \overline{a}) = L$, so that $L = \overline{k}_m$.

Since $\frac{\underline{k}_M+\overline{k}_m}{2} < \overline{k}_m$, there exists some $N_1 \in N_+$ such that

$$0 < \overline{k}_m - k_t < \overline{k}_m - \frac{\underline{k}_M + \overline{k}_m}{2},$$

and in particular $k_t > \frac{\underline{k}_M+\overline{k}_m}{2}$, for any $t \geq N_1$. Since $a_t \in [\underline{a}, \overline{a}]$ for any $t \geq 1$, this now implies that

$$Q^t(c, [\hat{s}, d]) \geq v(\{\overline{a}\})^t$$

for any $t \geq N_1$.

A symmetric construction will show the other desired inequality. We now generate $\{k_t, a_t\}_{t\in\mathbb{N}}$ by putting $(k_0, a_0) = d = (\overline{k}_M, \overline{a})$ and defining

$$k_{t+1} = k^*(k_t, a_t), \quad a_{t+1} = \underline{a}.$$

for any $t \geq 0$. By construction, $k_1 = \overline{k}_M$ and

$$k_{t+1} = k^*(k_t, \underline{a}) \quad \text{for any } t \geq 1.$$

Note that $\underline{k}_M < \overline{k}_M = k_1$. Suppose $\underline{k}_M < k_t$ for some $t \geq 1$. Then,

$$k_{t+1} = k^*(k_t, \underline{a}) > k^*(\underline{k}_M, \underline{a}) = \underline{k}_M,$$

so that $\{k_t\}_{t\in\mathbb{N}}$ is contained in the interval $(\underline{k}_M, \overline{k}_M]$. Since $k^*(k, \underline{a}) < k$ for any $k > \underline{k}_M$, it follows that

$$k_{t+1} = k^*(k_t, \underline{a}) > k_t$$

and therefore that $\{k_t\}_{t\in\mathbb{N}}$ is a strictly decreasing sequence with lower bound $\underline{k}_M$. This implies that there exists some limit $L \geq \underline{k}_M$ such that $k_t \searrow L$ as $t \to \infty$; taking limits on the chain of inequalities above shows us that $k^*(L, \underline{a}) = L$, so that $L = \underline{k}_M$.

Since $\underline{k}_M < \frac{\underline{k}_M+\overline{k}_m}{2}$, there exists some $N_2 \in N_+$ such that

$$k_t < \underline{k}_M$$

for any $t \geq N_2$. Since $a_t \in \{\underline{a}\}$ for any $t \geq 1$, this now implies that

$$Q^t(d, [c, \hat{s}]) \geq v(\{\underline{a}\})^t$$

for any $t \geq N_2$.

Defining $N = \max(N_1, N_2)$ and

$$\epsilon = \min\left(v(\{\overline{a}\})^N,\ v(\{\underline{a}\})^N\right) > 0,$$

it follows that

$$Q^N(c, [\hat{s}, d]) \geq \epsilon, \quad Q^N(d, [c, \hat{s}]) \geq \epsilon,$$

as we desired.

Q.E.D.

We have shown that Q is a Markov kernel on $(X \times Z, \mathcal{X} \bigotimes \mathcal{Z})$ has the Feller property, is monotonic, has stable set

$$[c, d] = [\underline{k}_m, \overline{k}_M] \times Z$$

such that

$$\lim_{t \to \infty} Q^t(s_0, [c, d]) = 1$$

for any $s_0 \in (0, \hat{k}] \times Z$, and that it satisfies the splitting condition on $[c, d]$. Therefore, there exists a stationary distribution μ^* of Q concentrated on $[c, d]$ such that

$$\delta_{s_0} Q^t \to \mu^*$$

weakly as $t \to \infty$ for any initial value $s_0 \in (0, \hat{k}] \times Z$. The stationary distribution μ^* is the stochastic counterpart to the deterministic steady state level of capital.

Crucially, note how this result differs from the stronger one that we proved in the section on the weak convergence of Markov chains. There, we showed that, under certain

assumptions, a Markov kernel had a unique stationary distribution to which any Markov chain converges weakly, regardless of the initial distribution. Here, we made the weaker claim that there exists some non-degenerate stationary distribution of Q concentrated on the stable set $[c,d]$ to which any Markov chain started at an initial value with non-zero capital stock converges weakly. This is because the stationary distribution is not unique in this case; since $k^*(0,a)=0$ for any $a \in Z$, any distribution μ_0 on the product space $(X,\mathcal{X})\bigotimes(Z,\mathcal{Z})$ such that

$$\mu_0(\{0\}\times Z)=1$$

is a stationary distribution of Q. This, however, is a degenerate case in that capital never evolves, and there is no economy to speak of.

One final remark is in order. We have established that

$$\mu^*([\underline{k}_m,\overline{k}_M]\times Z)=1$$

for the non-trivial stationary distribution μ^* of Q. However, it is possible to further narrow down the set on which μ^* is concentrated:

Lemma (Support of Stationary Distribution)
Consider the stochastic neoclassical growth model as given above, and assume that $k*(k,a)>0$ for any $(k,a)\in(0,\hat{k}]\times Z$. Let μ^* be the non-trivial stationary distribution of Q found above. Then,

$$\mu^*([\underline{k}_M,\overline{k}_m]\times Z)=1.$$

Proof) To see this, we first show that $[\underline{k}_M,\overline{k}_m]\times Z$ is a stable set for Q. For any $(k,a)\in[\underline{k}_M,\overline{k}_m]\times Z$, we can see that

$$k^*(k,a)\leq k^*(\overline{k}_m,\overline{a})=\overline{k}_m,$$

and similarly,

$$k^*(k,a)\geq k^*(\underline{k}_M,\underline{a})=\underline{k}_M,$$

so that $k^*(k,a) \in [\underline{k}_M, \overline{k}_m]$. Therefore,

$$Q((k,a), [\underline{k}_M, \overline{k}_m] \times Z) = 1,$$

and $[\underline{k}_M, \overline{k}_m] \times Z$ is stable for Q.

Now choose any $s_0 = (k_0, a_0) \in [\underline{k}_M, \overline{k}_m] \times Z$. We saw above that

$$\delta_{s_0} Q^t \to \mu^*$$

weakly as $t \to \infty$, and because $[\underline{k}_M, \overline{k}_m] \times Z$ is a stable set,

$$Q^t(s_0, [\underline{k}_M, \overline{k}_m] \times Z) = 1$$

for any $t \geq 1$. Since $[\underline{k}_M, \overline{k}_m] \times Z$ is a closed subset of $X \times Z$, the Portmanteau theorem tells us that

$$1 = \liminf_{t \to \infty} Q^t(s_0, [\underline{k}_M, \overline{k}_m] \times Z) \leq \mu^*([\underline{k}_M, \overline{k}_m] \times Z),$$

which implies that $\mu^*([\underline{k}_M, \overline{k}_m] \times Z) = 1$.

Q.E.D.

We can therefore see that the steady state distribution assigns all mass to capital stock between the $\underline{k}_M$, the largest fixed point of the worst realization of productivity, and $\overline{k}_m$, the smallest fixed point of the best realization of productivity.

5.2 The Real Business Cycle Model

In the consumption-savings model, we discussed how an infinitely lived rational consumer consumes and saves over time, assuming that he receives an exogenous stream of income. We expanded our perspective slightly with the stochastic neoclassical growth model, in which we studied the social planner's problem (and in doing so the competitive equilibrium allocation) of capital investment under stochastic productivity shocks. In either model, we have not discussed the role of labor supply.

In the real business cycle (RBC) model, we discuss this aspect of the macroeconomy that we have neglected so far. As in the stochastic neoclassical growth model, the RBC model assumes that business cycle fluctuations arise as the result of stochastic productivity shocks, and studies how labor supply decisions evolve in the midst of these fluctuations.

5.2.1 Model Assumptions

The assumptions of the RBC model are as follows:

Commodities There are many homogeneous households and firms. Each period, the household consumes the final good, supplies labor hours, and lends capital to the firm. The firm produces the final good using a combination of labor and capital, where the technology depends on the productivity at that period.

Technologies Capital takes values in $\mathbb{R}_+$, and labor in the compact interval $[0, \overline{L}]$. Here, $\overline{L}$ represents the maximum labor hours that can be supplied in each period; usually, it is taken to be 24 hours, or normalized to 1. The state space in this model is the space of capital and labor, and so is defined as $X = \mathbb{R}_+ \times [0, \overline{L}]$.

Productivity takes values in the compact interval $Z = [\underline{a}, \overline{a}] \subset \mathbb{R}$, with associated Borel σ-algebra $\mathcal{Z}$. We assume that productivity is a Markov chain that evolves according to the Markov kernel P. We further assume that P has the Feller property, that is, for any $\{z_n\}_{n \in N_+} \subset Z$ that converges to some $z \in Z$,

$$P(z_n, \cdot) \to P(z, \cdot) \quad \text{weakly as } n \to \infty.$$

We will once again be opting for the representation of the exogenous shock process via stochastic processes; let $\{A_t\}_{t\in\mathbb{N}}$ be the Markov chain taking values in $(Z, \mathcal{Z})$ with Markov kernel P, that represents the productivity process.

The firm produces final goods based on the production function

$$y_t = F(k_t, l_t, a_t),$$

where $F : X \times Z \to \mathbb{R}_+$ satisfies the usual conditions

$$F_k(\cdot, a) > 0, \quad F_{kk}(\cdot, a) < 0,$$

$$F_l(\cdot, a) > 0, \quad F_{ll}(\cdot, a) < 0$$

for marginal productivity for any $a \in Z$. The subscripts represent derivatives with respect to each input; we are basically assuming positive but diminishing marginal products for both capital and labor. We also impose the stronger assumption that $F(\cdot, a)$ is strictly concave on X.

Furthermore, the Inada conditions

$$\lim_{k\downarrow 0} F_k(\cdot, a) = +\infty, \quad \lim_{k\to\infty} F_k(\cdot, a) = 0$$

hold for any $a \in Z$. Once again, these assumptions imply the existence of a maximum maintainable and attainable level of capital $\bar{k} > 0$, so that we can restrict the state space as $X = [0, \bar{k}] \times [0, \overline{L}]$ without any loss of generality.

Finally, we assume that $F(k, l\cdot)$ is strictly increasing on Z for any $(k, l) \in X$; in other words the higher productivity, the greater the output given the same amount of inputs.

We assume that capital depreciates at a rate $\delta \in [0, 1]$ each period, so that investment at time t satisfies

$$k_{t+1} = i_t + (1 - \delta)k_t.$$

Finally, there is no government spending, nor is there a foreign sector, so that

output each period is fully spent as either consumption or investment:

$$y_t = c_t + i_t.$$

Preferences The representative household's flow utility function is given as $u : \mathbb{R} \times [0, \bar{L}] \to \mathbb{R}$, so that, once again, consuming negative values is allowed (for computational reasons). This time, the household's source of utility is not only consumption, but also the amount of labor it supplies. u is assumed to be twice continuously differentiable with derivatives

$$u_c(\cdot) > 0, \quad u_l(\cdot) < 0.$$

In other words, u is increasing in consumption but decreasing in labor. We also impose the assumption that u is jointly strictly concave in both arguments; this is the case, for instance, if utility is CRRA and separable in consumption and labor, so that

$$u(c, l) = \frac{c^{1-\theta}}{1-\theta} + \varphi \cdot \frac{(\bar{L} - l)^{1-\gamma}}{1-\gamma}.$$

The representative consumer's lifetime utilty under the consumption and labor supply path $\{(c_t, l_t)\}_{t \in \mathbb{N}}$ is

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \cdot u(c_t, l_t) \right],$$

where the expectation is taken conditional on the information contained in initial productivity.

5.2.2 Preliminary Model Analysis

As usual, the social planner's sequence problem is given as

$$\begin{aligned} \max_{\{(c_t,l_t)\}_{t\in\mathbb{N}}} \quad & \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\cdot u(c_t,l_t)\right] \\ \text{subject to} \quad & c_t\geq 0, \quad l_t\in[0,\bar{L}], \quad k_t\geq 0 \\ & c_t+k_{t+1}\leq F(k_t,l_t,a_t)+(1-\delta)k_t \quad \text{for any } t\in\mathbb{N}. \end{aligned}$$

The state-only formulation of the problem is

$$\begin{aligned} \max_{\{(k_t,l_t)\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty}\beta^t\mathbb{E}_0\left[u\left[F(k_t,l_t,a_t)+(1-\delta)k_t-k_{t+1},\ l_t\right]\right] \\ \text{subject to} \quad & l_t\in[0,\bar{L}], \\ & k_{t+1}\in[0,\ F(k_t,l_t,a_t)+(1-\delta)k_t] \quad \text{for any } t\in\mathbb{N}. \end{aligned}$$

Defining the instantaneous return function $U: X\times X\times Z\to\mathbb{R}$ and the constraint correspondence $\Gamma: X\times Z\to X$ as

$$U((k,l),(k',l'),a)=u\left[F(k,l',a)+(1-\delta)k-k',\ l'\right]$$

$$\Gamma((k,l),a)=\left\{(k',l')\in X\mid l'\in[0,\bar{L}], \quad k'\in\left[0,\ F(k,l',a)+(1-\delta)k\right]\right\}$$

for any $(k,l),(k',l')\in X$ and $a\in Z$, a feasible plan given the initial values (k_0,a_0) is a plan $\{\pi_t\}_{t\in\mathbb{N}}$ such that

$$\pi_0\in\Gamma((k_0,0),a_0)$$

$$\pi_t(a^t)\in\Gamma(\pi_{t-1}(a^{t-1}),a_t) \quad \text{for any } t\in\mathbb{N}, a^t\in Z^t,$$

where "initial" labor supply is given as $l_{-1}=0$, since neither the instantaneous return function nor the constraint correspondence depend on it.

Let $\Pi(k_0,a_0)$ be the set of all feasible plans given the initial values (k_0,a_0). The

planner's problem can finally be formulated as

$$\max_{\pi\in\Pi(k_0,a_0)} \quad U\left[(k_0,0),\pi_0,a_0\right]+\sum_{t=1}^{\infty}\beta^t\cdot\int_{Z^t}U\left[\pi_{t-1}(a^{t-1}),\pi_t(a^t),a_t\right]\,\mu^t(a_0,da^t).$$

The functional equation corresponding to the sequence problem becomes

$$V((k,l),a)=\sup_{(k',l')\in\Gamma((k,l),a)}\left[U((k,l),(k',l'),a)+\beta\cdot\int_Z V((k',l'),a')P(a,da')\right].$$

for any $(k,l)\in X$ and $a\in Z$.

Before moving onto checking the general assumptions, we first prove the following result:

Lemma (Well-behaved Constraint Correspondence)
Under the model assumptions above, let $\Gamma: X\times Z\to X$ be the constraint correspondence of the planner's problem defined as

$$\Gamma((k,l),a)=\left\{(k',l')\in X\mid l'\in[0,\bar{L}],\quad k'\in\left[0,\ F(k,l',a)+(1-\delta)k\right]\right\}$$

for any $(k,l)\in X$ and $a\in Z$. Then, Γ is a non-empty compact valued and continuous correspondence on $X\times Z$.

Proof) We first verify non-empty compact valuedness. Choose any $(k,l)\in X$ and $a\in Z$. Then, since $[0,\bar{L}]$ is non-empty and $F(k,\bar{L},a)+(1-\delta)k>0$, the interval $[0,F(k,\bar{L},a)+(1-\delta)k]$ is non-empty as well. It follows that $\Gamma((k,l),a)$ is a non-empty set. Now let $\{(k'_n,l'_n)\}_{n\in N_+}$ be a sequence in the set $\Gamma((k,l),a)$. Since this sequence takes values in the compact set X, there exists a subsequence $\{(k'_{n_m},l'_{n_m})\}_{m\in N_+}$ that converges to some point $(k^*,l^*)\in X$. By definition, $l^*\in[0,\bar{L}]$, and since

$$0\le k'_{n_m}\le F(k,l'_{n_m},a)+(1-\delta)k$$

for any $m\in N_+$ and F is a continuous function, taking $m\to\infty$ on both sides implies

$$0\le k^*\le F(k,l^*,a)+(1-\delta)k,$$

so that $(k^*, l^*) \in \Gamma((k,l),a)$. This shows us that $\Gamma((k,l),a)$ is sequentially compact and thus compact in the light of their equivalence on metric spaces.

Finally, we must verify that Γ is a continuous correspondence. Choose any $(k,l) \in X$ and $a \in Z$, and let $\{((k_n,l_n),a_n)\}_{n\in N_+}$ be a sequence in $X \times Z$ converginng to $((k,l),z)$. To first show upper hemicontinuity, let $\{(k'_n,l'_n)\}_{n\in N_+}$ be a convergent sequence with limit $(k^*,l^*) \in X$ such that $(k'_n,l'_n) \in \Gamma((k_n,l_n),a_n)$ for any $n \in N_+$. Then, since

$$0 \leq l'_n \leq \bar{L}$$

and

$$0 \leq k'_n \leq F(k_n,l'_n,a_n) + (1-\delta)k_n$$

for any $n \in N_+$, taking $n \to \infty$ for both inequalities yields

$$0 \leq l^* \leq \bar{L}, \quad 0 \leq k^* \leq F(k,l^*,a) + (1-\delta)k,$$

so that $(k^*,l^*) \in \Gamma((k,l),a)$. This proves that Γ is upper hemicontinuous at $((k,l),a)$.

Lower hemicontinuity is slightly trickier. First, fix any point $(k^*,l^*) \in \Gamma((k,l),a)$, and note that $l^* \in [0,\bar{L}]$ and $0 \leq k^* \leq F(k,l^*,a) + (1-\delta)k$. Define $l'_n = l^*$ for any $n \in N_+$, and define

$$k'_n = \min\left[F(k_n,l^*,a_n) + (1-\delta)k_n,\ k^*\right]$$

for any $n \in N_+$. Then, by construction, $l'_n \in [0,\bar{L}]$ and

$$0 \leq k'_n \leq F(k_n,l'_n,a_n) + (1-\delta)k_n$$

for any $n \in N_+$, so that $(k'_n,l'_n) \in \Gamma((k_n,l_n),a_n)$. Furthermore, since

$$\lim_{n\to\infty} \left(F(k_n,l^*,a_n) + (1-\delta)k_n\right) = F(k,l^*,a) + (1-\delta)k \geq k^*,$$

it follows that

$$\lim_{n\to\infty} k'_n = k^*.$$

Therefore, $\{(k'_n, l'_n)\}_{n\in N_+}$ is a sequence converging to (k^*, l^*) such that $(k'_n, l'_n) \in \Gamma((k_n, l_n), a_n)$ for any $n \in N_+$. This proves that Γ is lower hemicontinuous at $((k, l), a)$.

Q.E.D.

Checking the General Assumptions

We once again check whether the six general assumptions hold for this model.

A3 : **State Space and Space of Productivity Shocks**
The state space $X = [0, \bar{k}] \times [0, \bar{L}]$ is compact, as is the set Z on which the productivity shocks take values. This indicates that $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ are both complete and separable subspaces of euclidean space.

A4 : **Feller Property of Markov Kernel** P
P has the Feller property by assumption.

A5 : **Well-behaved Constraint Correspondence**
Note that the constraint correspondence takes on a slightly more complicated form in this model due to the inclusion of labor. Thus, we must painstakingly verify whether Γ is non-empty compact valued and continuous. This was the content of the preceding lemma.

A6 : **Well-behaved Instantaneous Return Function**
The return function $U : X \times X \times Z \to \mathbb{R}$ is continuous on $X \times X \times Z$, which is a compact set, so it is also bounded.

A7 : **Concavity Assumptions**

Fix $a \in Z$. Checking the concavity assumptions is also tricky due to the added state variable, but it is nevertheless still possible. Fix $a \in Z$ and define the function $h : X \times X \to [0, \bar{k}]$ as

$$h((k,l),(k',l')) = F(k,l',a) + (1-\delta)k - k'.$$

The Hessian of h at $((k,l),(k',l')) \in X \times X$ then becomes

$$D^2 h((k,l),(k',l')) = \begin{pmatrix} F_{kk}(k,l',a) & 0 & 0 & F_{kl}(k,l',a) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_{lk}(k,l',a) & 0 & 0 & F_{ll}(k,l',a) \end{pmatrix},$$

which is negative semidefinite owing to the joint concavity of F. Since

$$U((k,l),(k',l'),a) = u\left[h((k,l),(k',l')),l'\right],$$

where u is jointly concave in its two arguments, the mappings $((k,l),(k',l')) \mapsto h((k,l),(k',l'))$ and $((k,l),(k',l')) \mapsto l'$ are both concave, and the composition of concave functions is also concave, $U(\cdot,a)$ is concave on $X \times X$.

Fixing $a \in Z$ and $(k,l) \in X$, we must also show that $U((k,l),\cdot,a)$ is a strictly concave function on X. To this end, choose any distinct $x, x' \in X$ and $\lambda \in (0,1)$. Defining

$$c = F(k,x_2,a) + (1-\delta)k - x_1$$

$$c' = F(k,x_2',a) + (1-\delta)k - x_1',$$

$$\tilde{c} = F(k,\lambda \cdot x_2 + (1-\lambda)\cdot x_2',a) + (1-\delta)k - \left(\lambda \cdot x_1 + (1-\lambda)x_1'\right),$$

note that

$$\lambda c + (1-\lambda)c' = \lambda \cdot F(k,x_2,a) + (1-\lambda)\cdot F(k,x_2',a) + (1-\delta)k - \left(\lambda \cdot x_1 + (1-\lambda)x_1'\right)$$

$$< F\big(k, \lambda \cdot x_2 + (1-\lambda)\cdot x_2', a\big) + (1-\delta)k - \left(\lambda \cdot x_1 + (1-\lambda)x_1'\right) = \tilde{c}.$$

(Strict Concavity of F)

Therefore,

$$\begin{aligned}
\lambda \cdot U\left[(k,l), x, a\right] + (1-\lambda)\cdot U\left[(k,l), x', a\right] &= \lambda \cdot u\left[c, x_2\right] + (1-\lambda)\cdot u\left[c', x_2'\right] \\
&\leq u\left[\lambda c + (1-\lambda)c',\ \lambda x_2 + (1-\lambda)x_2'\right] \\
&\qquad \text{(Concavity of } u\text{)} \\
&< u\left[\tilde{c},\ \lambda x_2 + (1-\lambda)x_2'\right]. \\
&\qquad (u \text{ is strictly increasing in consumption}) \\
&= U\left[(k,l), \lambda x + (1-\lambda)x', a\right].
\end{aligned}$$

This proves that $U\big((k,l), \cdot, a\big)$ is strictly concave on X.

As for the constraint correspondence, fix $a \in Z$ and choose any

$$y \in \Gamma(x,a), \quad y' \in \Gamma(x',a)$$

for any $x, x' \in X$. Choose $\lambda \in [0,1]$.

By definition,

$$\begin{aligned}
y_2, y_2' &\in [0, \bar{L}], \\
0 \leq y_1 &\leq F(x_1, y_2, a) + (1-\delta)x_1, \\
0 \leq y_1' &\leq F(x_1', y_2', a) + (1-\delta)x_1'.
\end{aligned}$$

By the concavity of the production function in both inputs,

$$\begin{aligned}
0 \leq \lambda y_1 + (1-\lambda)y_1' &\leq \lambda \cdot F(x_1, y_2, a) + (1-\lambda)\cdot F(x_1', y_2', a) + (1-\delta)\left[\lambda x_1 + (1-\lambda)x_1'\right] \\
&\leq F\left[\lambda x_1 + (1-\lambda)x_1',\ \lambda y_2 + (1-\lambda)y_2',\ a\right] + (1-\delta)\left[\lambda x_1 + (1-\lambda)x_1'\right],
\end{aligned}$$

and

$$\lambda y_2 + (1-\lambda) y_2' \in [0, \bar{L}]$$

by the convexity of $[0, \bar{L}]$, so

$$\lambda y + (1-\lambda) y' \in \Gamma\left(\lambda x + (1-\lambda) x', a\right).$$

This proves that Γ is appropriately convex, as desired.

A9 : **Differentiability Assumptions**

For any $a \in Z$, $U(\cdot, \cdot, a)$ is twice continuously differentiable on the interior $X^o \times X^o$ of $X \times X$ by the twice continuous differentiability of the production and utility functions.

Preliminary Model Results

Now it follows from the results we proved in the previous chapter that:

1) **Existence and Uniqueness of Solution to Functional Equation**
There exists a unique continuous and bounded solution V^* to the functional equation:

$$V^*(k,l,a) = \sup_{\substack{l' \in [0,\bar{L}] \\ k' \in [0,\, F(k,l',a)+(1-\delta)k]}} \left[u[F(k,l',a) + (1-\delta)k - k',\, l'] + \beta \cdot \int_Z V^*(k',l',a') P(a, da') \right]$$

for any $(k,l) \in X$ and $a \in Z$. Furthermore, $V^*(\cdot, a)$ is concave on X for any $a \in Z$.

2) **Continuous and Single-valued Policy Correspondence**

There exists a continuous policy function $g = (k^*, l^*) : X \times Z \to X$ such that

$$V^*(k,l,a) = u[F(k,l^*(k,l,a),a) + (1-\delta)k - k^*(k,l,a),\ l^*(k,l,a)]$$

$$+ \beta \cdot \int_Z V^*(g(k,l,a),a')P(a,da')$$

for any $(k,l) \in X$ and $a \in Z$. Similarly, there exists a continuous optimal consumption policy funciton $c^* : X \times Z \to \mathbb{R}_+$ defined as

$$c^*(k,l,a) = F(k,l^*(k,l,a),a) + (1-\delta)k - k^*(k,l,a)$$

for any $(k,l) \in X$ and $a \in Z$.

3) **Envelope Theorem for Value Function**

For any $a \in Z$ and $(k,l) \in (0,\bar{k}) \times (0,\bar{L})$, assuming that $g(k,l,a) \in \Gamma(k,l,a)^o$, we have

$$\nabla_x V^*(k,l,a) = \nabla_{(k,l)} U\left[(k,l), g(k,l,a), a\right]$$

$$= \begin{pmatrix} u_c\left[c^*(k,l,a), l^*(k,l,a)\right] \cdot \left[F_k(k,l^*(k,l,a),a) + (1-\delta)\right] \\ 0 \end{pmatrix}.$$

This tells us that V^* is constant in labor; this retroactively means that we could have treated capital as our sole state variable from the beginning.

5.2.3 The Euler Equation

Assume that $g(k,l,a) \in \Gamma(k,l,a)^o$ for any $(k,l,a) \in X \times Z$. Then, by the first order conditions for maximization and the bounded convergence theorem (which allows us to exchange the order of integration and differentiation)

$$\begin{pmatrix} -u_c\left[c^*(k,l,a), l^*(k,l,a)\right] \\ u_c\left[c^*(k,l,a), l^*(k,l,a)\right] \cdot F_l(k, l^*(k,l,a), a) + u_l\left[c^*(k,l,a), l^*(k,l,a)\right] \end{pmatrix}$$

$$+ \beta \cdot \int_Z \nabla_x V^*(g(k,l,a), a') P(a, da') = O_{2\times 1}.$$

for any $(k,l,a) \in X \times Z$. Using the envelope theorem for the value function, we can see that the second equation above can be written as

$$u_c\left[c^*(k,l,a), l^*(k,l,a)\right] \cdot F_l(k, l^*(k,l,a), a) = -u_l\left[c^*(k,l,a), l^*(k,l,a)\right],$$

which represents the intratemporal substitution between consumption and labor. The first equation, meanwhile, yields the (somewhat more complicated) usual Euler equation

$$u_c\left[c^*(k,l,a), l^*(k,l,a)\right]$$

$$= \beta \cdot \int_Z u_c\left[c^*(g(k,l,a),a'), l^*(g(k,l,a),a')\right] \cdot \left[F_k(k^*(k,l,a), l^*(g(k,l,a),a'), a') + 1 - \delta\right] P(a, da')$$

The similarity will be apparent in a moment.

Given the stochastic process $\{A_t\}_{t\in\mathbb{N}}$ representing the productivity shocks and initial capital k_0, the optimal path of capital, labor and consumption $\{K_t\}_{t\in\mathbb{N}}$, $\{L_t\}_{t\in\mathbb{N}}$ and $\{C_t\}_{t\in\mathbb{N}}$ are defined recursively as

$$K_0 = k_0$$

$$K_{t+1} = k^*(K_t, L_t, A_t) \quad \text{for any } t \in \mathbb{N}$$

and

$$L_0 = l^*(k_0, 0, A_0)$$

$$L_t = l^*(K_t, L_{t-1}, A_t) \quad \text{for any } t \in \mathbb{N}$$

and

$$C_t = F(K_t, L_t, A_t) + (1-\delta)K_t - K_{t+1} \quad \text{for any } t \in \mathbb{N}.$$

The Euler equation can now be written as

$$u_c(C_t, L_t) \cdot F_l(K_t, L_t, A_t) = -u_l(C_t, L_t)$$

(Intratemporal Substitution of Consumption and Labor)

$$u_c(C_t, L_t) = \beta \cdot \mathbb{E}_t\left[u_c(C_{t+1}, L_{t+1}) \cdot [F_k(K_{t+1}, L_{t+1}, A_{t+1}) + 1 - \delta]\right]$$

(Intertemporal Substitution of Consumption)

where $\mathbb{E}_t[\cdot]$ is the conditional expectation given the information of the productivity shocks up to time t, that is, conditional on the σ-algebra generated by $A_0, \cdots, A_t$.

Chapter 6

Rational Expectations Equilibrium

So far, we have derived equilibrium allocations via the social planner's problem. The rationale for investigating the social planner's problem instead of solving directly for the equilibrium allocation is the first fundamental theorem of welfare economics, which tells us that any competitive equilibrium allocation is Pareto efficient; and the additional result that any Pareto efficient allocation can be achieved via the social planner's problem under the appropriate weights. In economies populated by homogeneous households, the setup of the social planner's problem is invariant to the weights the social planner imposes on each household, and therefore the allocation achieved in the social planner's problem is equivalent to the competitive equilibrium allocation.

However, there are some cases in which we are interested in solving for equilibrium directly. First, and most obviously, we might sometimes be interested in studying the evolution of equilibrium prices, which are completely absent from the social planner's problem. In addition, if the assumptions of the first welfare theorem break down, then the equilibrium allocation is not necessarily equivalent to the allocation achieved in the social planner's problem, as is the case in, most notably, settings of monopolistic competition.

To compute for competitive equilibrium prices and allocations in a recursive setting, we need to impose additional assumptions on the prices and how agents form beliefs about macroeconomic aggregates. Following the framework introduced in Prescott and Mehra (1985), we investigate how the notion of recursive competitive equilibrium can be applied to a number of macroeconomic problems, including the basic asset pricing model of Lucas (1978), and the stochastic neoclassical growth model studied above.

6.1 The Lucas Asset Pricing Model

The Lucas asset pricing model, often called the Lucas tree model, is a rudimentary asset pricing model that includes two assets meant to represent equity (stocks) and debt (bonds) in a pure endowment economy.

6.1.1 Model Assumptions

The model assumptions are given as follows:

Commodities There are many homogeneous consumers populating the economy with a mass of 1 and indexed by values in the continuum $[0,1]$. There is one final good from which the consumers derive utility, and two assets.

Assets We assume that the economy is endowed with a "tree" that yields a stream of uncertain payoffs $\{y_t\}_{t\in\mathbb{N}}$ in terms of the final good. At each time t, the time t payoff y_t is entirely consumed in that period. The payoffs take values in the compact interval $Z = [\underline{y}, \overline{y}] \subset \mathbb{R}$, where $\underline{y} > 0$, and assume that payoffs form a Markov chain with kernel P. We further assume (as usual) that P has the eller property, and let $\{Y_t\}_{t\in\mathbb{N}}$ be the stochastic process taking values in $(Z, \mathcal{Z})$ representing the payoff process.

At time t, each consumer holds n_t shares of the tree, and receives payoff $n_t y_t$. We can think of $n_t y_t$ as the dividends paid to the consumer with n_t shares. We normalize the total number of shares in the economy to 1, so that the shares of all consumers sum to 1. Every household has 1 share of the tree in the initial period.

In addition, consumers can hold a bond to finance their debt. Holding a unit of the bond at time t nets the consumer $1 + r_{t+1}$ units of the final good at time $t+1$. There is a net zero supply of bonds at each time period, and households start off with having issued no debt.

The state variables in this model are the share of stock n_t and the debt b_t that the household holds in each period. Therefore, the state space is defined as $X = [n_L, n_H] \times [-b_L, b_H]$, where $n_L > 0$ is an exogenous lower limit for the shares of stock that the household can hold and n_H is an upper limit; $-b_L$ and b_H are defined analogously for some $b_L > 0$.

Preferences The representative household's flow utility function is given as $u : \mathbb{R} \to \mathbb{R}$, as usual. Furthermore, we assume that u is twice continuously differentiable with derivatives

$$u'(\cdot) > 0, \quad u''(\cdot) < 0$$

on $\mathbb{R}$, so that households face positive but diminishing marginal utility.

The representative consumer's lifetime utiliy under the consumption path $\{c_t\}_{t \in \mathbb{N}}$ is given as

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \right],$$

where the expectation is taken conditional on the initial payoff $y_0 \in Z$ of the tree.

6.1.2 Competitive Equilibrium

A competitive (Radner) equilibrium in this economy refers to a collection of prices $\{r_t, p_t\}_{t\in\mathbb{N}}$ and an allocation $\{(c_t, n_t, b_t)\}_{t\in\mathbb{N}}$ such that, for any stream $\{y_t\}_{t\in\mathbb{N}}$ of payoffs, we have

i) **Utility Maximization**
The representative household solves the following maximization problem:

$$\begin{aligned} \max_{\{(c_t,n_t,b_t)\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot \mathbb{E}_0\left[u(c_t)\right] \\ \text{subject to} \quad & c_t \geq 0,\ b_{t+1} \in [-b_L, b_H],\ n_{t+1} \in [-n_L, n_H] \\ & c_t + b_{t+1} + p_t n_{t+1} \leq p_t n_t + n_t y_t + (1+r_t) b_t \quad \text{for any } t \in \mathbb{N}. \end{aligned}$$

Interpreting the flow budget constraint, the representative household's income at time t is the sum of $p_t n_t$, from selling his n_t shares of equity, $y_t n_t$, from the dividends from holding n_t shares of equity from time $t-1$ to t, and $(1+r_t)b_t$, from the interest accrued from issuing b_t units of debt at time $t-1$. This income is used to either consume the final good (c_t), invest in equity (n_{t+1}), or issue debt (b_{t+1}).

ii) **Market Clearing**
The market clears at each time t, that is,

$$\underline{b}_t = b_t = 0 \qquad \text{(Bond Market Clearing)}$$

$$\underline{n}_t = n_t = 0 \qquad \text{(Stock Market Clearing)}$$

$$y_t = c_t. \qquad \text{(Goods Market Clearing)}$$

6.1.3 Macroeconomic Aggregates and Asset Prices

The first step into obtaining a recursive representation of this sort of competitive equilibrium is to obtain the recursive formulation of the household's problem. As is, the prices r_t and p_t of the assets at time t might depend on the entirety of the payoff history $y_0, \cdots, y_t$ up to time t, which prevents the household's problem from admitting a recursive representation. This is where our last assumption comes in; the prices depend only on the current payoff y_t, and in addition only on the current market aggregates $\underline{b}_t$ and $\underline{n}_t$. Crucially, the prices depend only on macroeconomic aggregates, which allows us to maintain the setting that individual households are price takers. Formally, our assumption can be stated as follows:

Prices We assume that the interest rate r_t for the bond and the price p_t of a share of equity are functions of the current payoff y_t and the aggregate bond supply and shares $\underline{b}_t$ and $\underline{n}_t$. In other words, we assume that there exist continuous and bounded functions $\mathcal{R}, \mathcal{P} : Z \to \mathbb{R}$ such that $r_t = \mathcal{R}(y_t)$ and $p_t = \mathcal{P}(y_t)$ for any $t \in \mathbb{N}$. Here, we can let the price functions only depend on the current payoff because the macroeconomic aggregates in this model are fixed at equilibrium so that $\underline{b}_t = 0$ and $\underline{n}_t = 1$ for any $t \in \mathbb{N}$.

Furthermore, we assume that there exists some maximum interest rate $M > 0$ such that the function $\mathcal{R}$ takes values in $[0, M]$, and choose the lower bound $-b_L$ for debt such that

$$b_L = \sum_{t=0}^{\infty} \frac{1}{(1+M)^t} n_L \underline{y} = \frac{M+1}{M} \cdot n_L \underline{y} > 0.$$

In other words, b_L is the natural borrowing limit for the household in that b_L is the maximum amount of borrowing that the household would be able to pay back if it faced the worst possible scenario forever, that is, the highest interest rate M and the lowest payoff $\underline{y}$ with the lowest share of stock n_L.

Finally, and this is a technical condition that will be used later, we assume that $\mathcal{P}(y) > 0$ for any $y \in Z$ that is, the price of a share of equity is always positive.

As mentioned above, a core feature of the Lucas asset pricing model is that the aggregate

state variables $\underline{b}_t$ and $\underline{n}_t$ take especially simple forms at equilibrium, with values equal to 0 and 1, respectively. This means that we do not need to specify laws of motion for macroeconomic aggregates for this model, and in particular that the household's value function does not depend on these macroeconomic aggregates at this time. This is an assumption that we relax when we encounter more sophisticated models where the laws of motion for the macroeconomic aggregates do matter; one example is the stochastic neoclassical growth model, which we address in the subsequent section.

For now, note that a competitive equilibrium now refers to a pair of price functions $\mathcal{R}, \mathcal{P} : Z \to \mathbb{R}$ and an allocation $\{(c_t, n_t, b_t)\}_{t \in \mathbb{N}}$ such that, for any stream $\{y_t\}_{t \in \mathbb{N}}$ of payoffs, we have

i) **Utility Maximization**
The representative household solves the following maximization problem:

$$\max_{\{(n_t, b_t)\}_{t \in \mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot \mathbb{E}_0 \left[u \left[(\mathcal{P}(y_t) + y_t) n_t + (1 + \mathcal{R}(y_t)) b_t - b_{t+1} - \mathcal{P}(y_t) \cdot n_{t+1} \right] \right]$$

$$\text{subject to} \quad (n_{t+1}, b_{t+1}) \in X,$$

$$b_{t+1} + \mathcal{P}(y_t) \cdot n_{t+1} \in \left[-b_L + \mathcal{P}(y_t) \cdot n_L,\ (\mathcal{P}(y_t) + y_t) n_t + (1 + \mathcal{R}(y_t)) b_t \right] \quad \text{for any } t \in \mathbb{N}.$$

ii) **Market Clearing**
The market clears at each time t: $\underline{b}_t = b_t = 0$, $\underline{n}_t = n_t = 0$ and $y_t = c_t$.

6.1.4 To the Recursive Representation

To solve for the representative household's problem, define the instantaneous return function $U : X \times X \times Z \to \mathbb{R}$ and the constraint correspondence $\Gamma : X \times Z \to X$ as

$$U((n, b), (n', b'), y) = u \left[(\mathcal{P}(y) + y) n + (1 + \mathcal{R}(y)) b - b' - \mathcal{P}(y) \cdot n' \right]$$

and

$$\Gamma(n,b,y)=\left\{(n',b')\in X\mid b'+\mathcal{P}(y)n'\in[-b_L+\mathcal{P}(y)n_L,\ (\mathcal{P}(y)+y)n+(1+\mathcal{R}(y))b]\right\}$$

for any $(n,b,y)\in X\times Z$ and $(n',b')\in X$. Given the initial value $y_0\in Z$ of the payoff, a feasible plan $\{\pi_t\}_{t\in\mathbb{N}}$ is defined as

$$\pi_0\in\Gamma(1,0,y_0)$$

$$\pi_t(y^t)\in\Gamma(\pi_{t-1}(y^{t-1}),y_t)\quad\text{for any } t\in\mathbb{N},y^t\in Z^t.$$

We denote the set of all feasible plans started from the initial payoff y_0 as $\Pi(y_0)$. Then, the household's problem can be written as

$$\max_{\pi\in\Pi(y_0)}\quad U\left[(1,0),\pi_0,y_0\right]+\sum_{t=0}^{\infty}\beta^t\cdot\int_{Z^t}U\left[\pi_{t-1}(y^{t-1}),\pi_t(y^t),y_t\right]\mu^t(y_0,dy^t),$$

and the dual functional equation is

$$V(n,b,y)=\sup_{(n',b')\in\Gamma(n,b,y)}\left[U((n,b),(n',b'),y)+\beta\cdot\int_Z V(n',b',y')P(y,dy')\right].$$

By the principle of optimality, a solution to the functional equation is also the value function of the sequence problem, and the plans generated by the corresponding optimal policy correspondence is (almost surely) exhaustive for optimal plans that solve the sequence problem.

6.1.5 Recursive Competitive Equilibrium

Therefore, a recursive competitive equilibrium (RCE) in the Lucas asset pricing model is a collection of:

- A value function $V^* : X \times Z \to \mathbb{R}$
- Policy functions $c^* : X \times Z \to \mathbb{R}_+$ and $a^* = (n^*, b^*) : X \times Z \to X$ of consumption and asset holdings
- Price functions $\mathcal{R}, \mathcal{P} : Z \to \mathbb{R}$
- Conjectures for macroeconomic aggregates $\underline{\mathrm{N}} = 1$ and $\underline{\mathrm{B}} = 1$

that satisfy the following conditions:

i) **Household Optimality**
The representative household optimizes given their conjectures for the aggregate state variables. Specifically, V^* solves the functional equation

$$V(n,b,y) = \sup_{(n',b') \in \Gamma(n,b,y)} \left[U((n,b),(n',b'),y) + \beta \cdot \int_Z V(n',b',y') P(y,dy') \right],$$

with a^* being the corresponding policy function and c^* defined as

$$c^*((n,b),y) = (\mathcal{P}(y)+y)n + (1+\mathcal{R}(y))b - \mathcal{P}(y)n^*((n,b),y) - b^*((n,b),y).$$

ii) **Belief Consistency**
The representative household's beliefs are correct, so that, for any $t \in \mathbb{N}$,

$$a^*((n,b),y) = (\underline{\mathrm{N}}, \underline{\mathrm{B}}) = (1,1)$$

for any $((n,b),y) \in X \times Z$.

iii) **Market Clearing**
All markets clear, so that

$$c^*((n,b),y) = y$$

for any $((n,b),y) \in X \times Z$.

6.1.6 Solving for RCE

It now remains to solve for the RCE in this model. To this end, we once again have to check all the general assumptions. Again, because this model has two state variables, it takes a bit of work to verify that the constraint correspondence is non-empty, compact valued and continuous.

Lemma (Well-behaved Constraint Correspondence)
Under the model assumptions above, let $\Gamma : X \times Z \to X$ be the constraint correspondence of the planner's problem defined as

$$\Gamma(n,b,y) = \left\{ (n',b') \in X \mid b' + \mathcal{P}(y)n' \in [-b_L + \mathcal{P}(y)n_L,\ (\mathcal{P}(y)+y)n + (1+\mathcal{R}(y))b] \right\}$$

for any $(n,b,y) \in X \times Z$. Then, Γ is a non-empty compact valued and continuous correspondence on $X \times Z$.

Proof) We first verify non-empty compact-valuedness; choose any $(n,b,) \in X \times Z$. Since $[-n_L, n_H]$ is non-empty valued, and

$$\begin{aligned}
\mathcal{P}(y)(n-n') + n \cdot y + (1+\mathcal{R}(y))b &\geq \mathcal{P}(y)(n_L - n') + n_L \cdot \underline{y} - (1+M)b_L \\
&= \mathcal{P}(y)(n_L - n') + n_L \cdot \underline{y} - \frac{1}{M} n_L \underline{y} \\
&= \frac{M-1}{M} n_L \underline{y} + \mathcal{P}(y)(n_L - n') \\
&\geq -\frac{M+1}{M} n_L \underline{y} + \mathcal{P}(y)(n_L - n') \\
&= -b_L + \mathcal{P}(y)(n_L - n')
\end{aligned}$$

by the way we set the natural borrowing limit as $b_L = \frac{M+1}{M} n_L \underline{y}$, $\Gamma(n,b,y)$ is a non-empty set. To see that it is compact, let $\{(n'_k, b'_k)\}_{k \in N_+}$ be a convergent

sequence in $\Gamma(n,b,y)$ with limit $(n',b') \in X$. Then, since

$$-b_L + \mathcal{P}(y) n_L \leq b'_k + \mathcal{P}(y) \cdot n'_k \leq (\mathcal{P}(y) + y)\, n + (1 + \mathcal{R}(y)) b$$

for any $k \in N_+$, taking $k \to \infty$ on both sides shows us that

$$-b_L + \mathcal{P}(y) n_L \leq b' + \mathcal{P}(y) \cdot n' \leq (\mathcal{P}(y) + y)\, n + (1 + \mathcal{R}(y)) b,$$

so that $(n',b') \in \Gamma(n,b,y)$. By implication, $\Gamma(n,b,y)$ is sequentially compact, and, owing to the equivalence of sequential compactness and compactness in metric spaces, compact.

Now choose some $(n,b,y) \in X \times Z$ and a sequence $\{(n_k, b_k, y_k)\}_{k \in N_+}$ in $X \times Z$ converging to (n,b,y). Choose a sequence $\{(n'_k, b'_k)\}_{k \in N_+}$ such that $(n'_k, b'_k) \in \Gamma(n_k, b_k, y_k)$ for any $k \in N_+$ that converges to some $(n',b') \in X$. Then, because

$$-b_L + \mathcal{P}(y_k) n_L \leq b'_k + \mathcal{P}(y) \cdot n'_k \leq (\mathcal{P}(y_k) + y_k)\, n_k + (1 + \mathcal{R}(y_k)) b_k$$

for any $k \in N_+$, and $\mathcal{P}(\cdot)$ and $\mathcal{R}(\cdot)$ are continuous functions on Z, taking $k \to \infty$ on both sides shows us that

$$-b_L + \mathcal{P}(y) n_L \leq b' + \mathcal{P}(y) \cdot n' \leq (\mathcal{P}(y) + y)\, n + (1 + \mathcal{R}(y)) b,$$

and therefore that $(n',b') \in \Gamma(n,b,y)$. This proves that Γ is upper hemicontinuous at (n,b,y).

As before, proving lower hemicontinuity is slightly more difficult. This time, we first choose $(n',b') \in \Gamma(n,b,y)$. We then define the sequence $\{(n'_k, b'_k)\}_{k \in N_+}$ as $n'_k = n'$ for any $k \in N_+$, and

$$b'_k = \min\left[(\mathcal{P}(y_k) + y_k)\, n_k - \mathcal{P}(y_k) \cdot n' + (1 + \mathcal{R}(y_k)) b_k,\ \ b'\right]$$

for any $k \in N_+$. Note that

$$\lim_{k \to \infty} \left[(\mathcal{P}(y_k) + y_k)\, n_k - \mathcal{P}(y_k) \cdot n' + (1 + \mathcal{R}(y_k)) b_k\right]$$

$$= (\mathcal{P}(y)+y)\,n - \mathcal{P}(y)\cdot n' + (1+\mathcal{R}(y))b > b',$$

where the last inequality follows because $\mathcal{P}(y)n' > 0$ (we finally use the technical condition introduced earlier!). It follows that, there exists some $N \in N_+$ such that, for any $k \geq N$, we hae

$$b'_k = b' < (\mathcal{P}(y_k)+y_k)\,n_k - \mathcal{P}(y_k)\cdot n' + (1+\mathcal{R}(y_k))b_k.$$

Now, for any $k \geq N$,

$$b'_k = b' \in [-b_L, b_H]$$

and

$$\begin{aligned} -b_L + \mathcal{P}(y_k)(n_L - n'_k) &\leq -b_L \\ &\leq b'_k \\ &\leq (\mathcal{P}(y_k)+y_k)\,n_k - \mathcal{P}(y_k)\cdot n'_k + (1+\mathcal{R}(y_k))b_k, \end{aligned}$$

where we used the fact that $n'_k = n'$, so that $(n'_k, b'_k) \in \Gamma(n_k, b_k, y_k)$. Finally, since $b'_k = b'$ for any $k \geq N$, it follows that

$$\lim_{k\to\infty} (n'_k, b'_k) = (n', b').$$

We have thus found a sequence $\{(n'_k, b'_k)\}_{k\in N_+}$ such that $(n'_k, b'_k) \in \Gamma(n_k, b_k, y_k)$ for any k large enough and which converges to (n', b'). It follows that Γ is lower hemicontinuous at (n, b, y).

Q.E.D.

It is now easy work to check the general assumptions one by one:

A3 : **State Space and Space of Productivity Shocks**
The state space $X = [n_L, n_H] \times [-b_L, b_H]$ is compact, as is the set Z on which the payoffs take values. This indicates that $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ are both complete and separable subspaces of euclidean space.

A4 : **Feller Property of Markov Kernel** P

P has the Feller property by assumption.

A5 : **Well-behaved Constraint Correspondence**

The preceding lemma shows us that Γ is a non-empty compact valued and continuous correspondence.

A6 : **Well-behaved Instantaneous Return Function**

The return function $U : X \times X \times Z \to \mathbb{R}$ is continuous on $X \times X \times Z$, which is a compact set, so it is also bounded.

A7 : **Concavity Assumptions**

Fix $y \in Z$, and define the function $h : X \times X \to \mathbb{R}$ as

$$h((n,b),(n',b')) = u\left[(\mathcal{P}(y)+y)\,n+(1+\mathcal{R}(y))b-b'-\mathcal{P}(y)\cdot n'\right]$$

for any $(n,b),(n',b') \in X$. By design, h is a linear function on $X \times X$ and trivially concave. Since $U(\cdot,y) = u \circ h$ and the composition of concave functions is concave, $U(\cdot,y)$ is a concave function on $X \times X$.

Fixing $y \in Z$ and $(n,b) \in X$, we must also show that $U((n,b),\cdot,y)$ is a strictly concave function on X. To this end, choose any distinct $x, x' \in X$ and $\lambda \in (0,1)$. Defining

$$c = (\mathcal{P}(y)+y)\,n+(1+\mathcal{R}(y))b-x_2-\mathcal{P}(y)\cdot x_1$$

$$c' = (\mathcal{P}(y)+y)\,n+(1+\mathcal{R}(y))b-x_2'-\mathcal{P}(y)\cdot x_1'$$

$$\tilde{c} = (\mathcal{P}(y)+y)\,n+(1+\mathcal{R}(y))b-\left(\lambda x_2+(1-\lambda)x_2'\right)-\mathcal{P}(y)\left(\lambda x_1+(1-\lambda)x_1'\right),$$

we can see that

$$\tilde{c} = \lambda \cdot c+(1-\lambda)c'.$$

It follows from the strict concavity of u that

$$U\left[(n,b),\lambda x+(1-\lambda)x',y\right]=u(\tilde{c})$$
$$>\lambda\cdot u(c)+(1-\lambda)u(c')=\lambda\cdot U\left[(n,b),x,y\right]+(1-\lambda)\cdot U\left[(n,b),x',y\right],$$

and thus we have established our desired strict concavity result.

As for the constraint correspondence, fix $y\in Z$ and choose any

$$w\in\Gamma(x,y),\quad w'\in\Gamma(x',y)$$

for any $x,x'\in X$. Choose $\lambda\in[0,1]$. Note that, by definition, $w,w'\in X$ and

$$-b_L+\mathcal{P}(y)\cdot n_L\leq w_2+\mathcal{P}(y)\cdot w_1\leq(\mathcal{P}(y)+y)\,x_1+(1+\mathcal{R}(y))\,x_2$$

and

$$-b_L+\mathcal{P}(y)\cdot n_L\leq w_2'+\mathcal{P}(y)\cdot w_1'\leq(\mathcal{P}(y)+y)\,x_1'+(1+\mathcal{R}(y))\,x_2'.$$

By the linearity of all the terms involved, we have

$$-b_L+\mathcal{P}(y)\cdot n_L\leq\left(\lambda w_2+(1-\lambda)w_2'\right)+\mathcal{P}(y)\cdot\left(\lambda w_1+(1-\lambda)w_1'\right)$$
$$\leq(\mathcal{P}(y)+y)\left(\lambda x_1+(1-\lambda)x_1'\right)$$
$$+(1+\mathcal{R}(y))\left(\lambda x_2+(1-\lambda)x_2'\right).$$

Since X is convex and thus $\lambda w+(1-\lambda)w'\in X$, by definition

$$\lambda w+(1-\lambda)w'\in\Gamma(\lambda x+(1-\lambda)x',y),$$

so that Γ is convex in the desired sense.

A9 : **Differentiability Assumptions**

For any $y \in Z$, $U(\cdot,\cdot,y)$ is twice continuously differentiable on the interior $X^o \times X^o$ of $X \times X$ by the twice continuous differentiability of the utility function.

Now it follows from the results we proved in the previous chapter that:

1) **Existence and Uniqueness of Solution to Functional Equation**
There exists a unique continuous and bounded solution V^* to the functional equation:

$$V^*(n,b,y) = \sup_{(n',b') \in \Gamma(n,b,y)} \left[U\left[(n,b),(n',b'),y\right] + \beta \cdot \int_Z V^*(n',b',y') P(y,dy') \right]$$

for any $(n,b,y) \in X \times Z$. Furthermore, $V^*(\cdot,y)$ is concave on X for any $y \in Z$.

2) **Continuous and Single-valued Policy Correspondence**
There exists a continuous policy function $a^* = (n^*, b^*) : X \times Z \to X$ such that

$$V^*(n,b,y) = u[(\mathcal{P}(y)+y)\,n + (1+\mathcal{R}(y))\,b - b^*(n,b,y) - \mathcal{P}(y) \cdot n^*(n,b,y)]$$

$$+ \beta \cdot \int_Z V^*(a^*(n,b,y),y') P(y,dy')$$

for any $(n,b,y) \in X \times Z$. Similarly, there exists a continuous optimal consumption policy funciton $c^* : X \times Z \to \mathbb{R}_+$ defined as

$$c^*(n,b,y) = (\mathcal{P}(y)+y)\,n + (1+\mathcal{R}(y))\,b - b^*(n,b,y) - \mathcal{P}(y) \cdot n^*(n,b,y)$$

for any $(n,b,y) \in X \times Z$.

3) **Envelope Theorem for Value Function**
For any $y \in Z$ and $(n,b) \in (n_L, n_H) \times (-b_L, b_H)$, assuming that $a^*(n,b,y) \in \Gamma(n,b,y)^o$,

we have

$$\boldsymbol{\nabla}_x V^*(n,b,y) = \boldsymbol{\nabla}_{(n,b)} U\left[(n,b), a^*(n,b,y), y\right] = \begin{pmatrix} u'\left[c^*(n,b,y)\right] \cdot \left(\mathcal{P}(y) + y\right) \\ u'\left[c^*(n,b,y)\right] \cdot \left(1 + \mathcal{R}(y)\right) \end{pmatrix}.$$

Assume that $a^*(n,b,y) \in \Gamma(n,b,y)^o$ for any $(n,b,y) \in X \times Z$. Then, by the first order conditions for maximization and the bounded convergence theorem, which allows us to interchange integration and differentiation, we obtain the equations

$$-u'\left[c^*(n,b,y)\right] \begin{pmatrix} \mathcal{P}(y) \\ 1 \end{pmatrix} + \beta \cdot \int_Z \boldsymbol{\nabla}_x V^*(a^*(n,b,y), y') P(y, dy') = O_{2\times 1}.$$

Together with the envelope theorem for the value function, we obtain our two Euler equations:

$$u'\left[c^*(n,b,y)\right] \cdot \mathcal{P}(y) = \beta \cdot \int_Z u'\left[c^*(g(n,b,y),y')\right] \cdot \left(\mathcal{P}(y') + y'\right) P(y, dy')$$

and

$$u'\left[c^*(n,b,y)\right] = \beta \cdot \int_Z u'\left[c^*(g(n,b,y),y')\right] \left(1 + \mathcal{R}(y')\right) P(y, dy').$$

6.2 The Stochastic Neoclassical Growth Model Revisited

Solving for the equilibrium allocation in the stochastic neoclassical growth model directly will give us more of a sense of what recursive competitive equilibrium entails.

6.2.1 Model Assumptions

We largely retain the same assumptions as in the Brock and Mirman model, tweaking only a couple of areas to accommodate the decentralized setting:

Commodities There are homogeneous households and firms of measure 1, indexed on the unit interval $[0, 1]$. The household owns capital, which it rents out to firms, and supplies the firm with labor hours chosen from in the unit interval $[0, 1]$. Each period, it decides how much of its income to spend on consumption, and how much to invest in capital. For simplicity, capital is measured in units of the final good, and the price of the final good and capital are equal and normalized to 1.

Meanwhile, the firm produces the final good using a combination of labor and capital, where the technology depends on the productivity at that period.

Technologies Capital takes values in the state space $X = \mathbb{R}_+$, and productivity takes values in the compact interval $Z = [\underline{a}, \overline{a}] \subset \mathbb{R}$, with associated Borel σ-algebra $\mathcal{Z}$. We assume that productivity is a Markov chain that evolves according to the Markov kernel P. We further assume that P has the Feller property. Let $\{A_t\}_{t\in\mathbb{N}}$ be the Markov chain taking values in $(Z, \mathcal{Z})$ with Markov kernel P, that represents the productivity process.

The firm produces final goods based on the production function

$$y_t = F(k_t, L_t, a_t),$$

where $F : \mathbb{R}_+ \times \mathbb{R}_+ \times Z \to \mathbb{R}_+$ satisfies the usual conditions for marginal products: for any $a \in Z$,

$$F_k(\cdot, a) > 0, \quad F_l(\cdot, a) > 0$$

and $F(\cdot, a)$ is strictly concave. $F(\cdot, a)$ is also homogeneous of degree 1, or exhibits constant returns to scale, for any $a \in Z$. This allows us to apply Euler's theorem to conclude that

$$F(k, l, a) = F_k(k, l, a)k + F_l(k, l, a)l$$

for any $(k, l, a) \in \mathbb{R}^2_+ \times Z$.

Unlike in our previous analsysis of Brock and Mirman's model, this time production explicitly depends on labor. However, we will for the most part be working with the simpler production function in which labor has been normalized; specifically, define

$$f(k, a) = F(k, 1, a)$$

for any $(k, a) \in \mathbb{R}_+ \times Z$. Then,

$$\frac{\partial F(k, 1, a)}{\partial k} = \frac{\partial f(k, a)}{\partial k},$$

so that the assumptions above imply that

$$\frac{\partial f(\cdot, a)}{\partial k} > 0, \quad \frac{\partial^2 f(\cdot, a)}{\partial k^2} < 0,$$

which are the same assumptions that we imposed in the earlier version of the model. As in that version, we also assume the Inada conditions

$$\lim_{k \downarrow 0} \frac{\partial f(\cdot, a)}{\partial k} = +\infty, \quad \lim_{k \to \infty} \frac{\partial f(\cdot, a)}{\partial k} = 0$$

hold for any $a \in Z$. By standard arguments, there exists a maximum maintainable and attainable level of capital $\bar{k} > 0$, so that we can restrict the state space as $X = [0, \bar{k}]$ without any loss of generality.

Furthermore, we assume that $f(k,\cdot)$ is strictly increasing on Z for any $k \in X$; in other words the higher productivity, the greater the output given the same amount of inputs.

We assume that capital depreciates at a rate $\delta \in [0,1]$ each period, so that investment at time t satisfies

$$k_{t+1} = i_t + (1-\delta)k_t.$$

Finally, there is no government spending, nor is there a foreign sector, so that output each period is fully spent as either consumption or investment:

$$y_t = c_t + i_t.$$

Preferences The representative household's flow utility function is given as $u : \mathbb{R} \to \mathbb{R}$, so that, once again, consuming negative values is allowed (for computational reasons). u is assumed to be twice continuously differentiable on $\mathbb{R}$ with derivatives

$$u'(\cdot) > 0, \quad u''(\cdot) < 0$$

on $\mathbb{R}$, so that u is strictly concave on $\mathbb{R}$.

The representative consumer's lifetime utilty under the consumption path $\{c_t\}_{t\in\mathbb{N}}$ is

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \cdot u(c_t)\right],$$

where the expectation is taken conditional on the information contained in initial productivity.

6.2.2 Competitive Equilibrium

A competitive (Radner) equilibrium in this economy is a collection of prices $\{w_t, R_t\}_{t\in\mathbb{N}}$ and allocations $\{(c_t, l_t^h, k_t^h)\}_{t\in\mathbb{N}}$, $\{(l_t^f, k_t^f)\}_{t\in\mathbb{N}}$ such that, for any stream $\{a_t\}_{t\in\mathbb{N}}$ of productivity shocks, we have

i) **Utility Maximization**
The representative household solves the following maximization problem:

$$\begin{aligned} \max_{\{(c_t,l_t^h,k_t^h)\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot \mathbb{E}_0\left[u(c_t)\right] \\ \text{subject to} \quad & c_t \geq 0,\ l_t^h \in [0,1],\ k_t^h \in [0,\bar{k}] \\ & c_t + k_{t+1}^h \leq w_t l_t^h + (1+R_t)k_t^h \quad \text{for any } t \in \mathbb{N}. \end{aligned}$$

Interpreting the flow budget constraint, the representative household's wealth at time t s the sum of labor income $w_t l_t^h$, the income from renting out capital $R_t k_t^h$, and the amount of capital k_t^h it holds at time t. This wealth is used to either consume the final good or invest in more capital goods.

ii) **Profit Maximization**
For any $t \in \mathbb{N}$, the representative firm solves the following maximization problem:

$$\max_{k_t^f, l_t^f} \quad \left[F(k_t^f, l_t^f, a_t) - w_t l_t^f - (R_t + \delta)k_t^f\right].$$

Here, the marginal cost from renting a unit of capital is the rental cost R_t plus the amount of capital δ that has depreciated during the course of using this rented unit of capital.

iii) **Market Clearing**
The market clears at any time t, that is,

$$\underline{k}_t = k_t^f = k_t^h \qquad \text{(Capital Market Clearing)}$$

$$\underline{l}_t = l_t^f = l_t^h \qquad \text{(Labor Market Clearing)}$$

$$y_t = F(\underline{k}_t, \underline{l}_t, a_t) = c_t + \underline{k}_{t+1} - (1-\delta)\underline{k}_t. \qquad \text{(Goods Market Clearing)}$$

6.2.3 Macroeconomic Aggregates and Prices

As in the Lucas asset pricing model, to reframe the household's problem in a recursive manner, we need to assume first that the factor prices w_t and R_t depend only on the productivity shock a_t at time t; and in addition that they are determined only by the aggregate levels of labor and capital. Since $l_t^h = 1$ for any $t \in \mathbb{N}$ in this model, we can ignore the dependence of the factor prices on labor, like we ignored the effect of aggregate debt and equity holdings on asset prices in the preceding model. Therefore, our assumption can be stated as follows:

Factor Prices We assume that the factor prices w_t and R_t are solely functions of the current productivity shock a_t and the aggregate capital stock $\underline{\mathrm{k}}_t$. In other words, there exist continuous functions $\mathcal{W}, \mathcal{R} : X \times Z \to \mathbb{R}_+$ such that $w_t = \mathcal{W}(\underline{\mathrm{k}}_t, a_t)$ and $R_t = \mathcal{R}(\underline{\mathrm{k}}_t, a_t)$ for any $t \in \mathbb{N}$.

Unlike in the Lucas asset pricing model, the aggregate capital stock $\underline{\mathrm{k}}_t$ is not fixed for every $t \in \mathbb{N}$, but rather evolves according to the equation

$$\underline{\mathrm{k}}_{t+1} = \underline{\mathrm{i}}_t + (1 - \delta)\underline{\mathrm{k}}_t,$$

where $\underline{\mathrm{i}}_t$ is aggregate investment. Thus, we make the following assumption about aggregate investment and, in large, the law of motion governing the aggregate state variable:

Law of Motion We assume that there exists a continuous function $\varphi : X \times Z \to X$ such that

$$\underline{\mathrm{k}}_{t+1} = \varphi(\underline{\mathrm{k}}_t, a_t)$$

for any $t \in \mathbb{N}$.

Similarly, we assume that there exists a continuous function $\hat{\varphi} : X \times Z \to X$ that represents the household's perceived aggregate law of motion. That is, the household believes that the aggregate state evolves according to the equation

$$\underline{\mathrm{k}}_{t+1} = \hat{\varphi}(\underline{\mathrm{k}}_t, a_t)$$

Under these assumptions, we can now redefine comptetitive equilibrium as a pair of price functions $\mathcal{W}, \mathcal{R} : X \times Z \to \mathbb{R}_+$, an aggregate law of motion $\varphi : X \times Z \to X$, a perceived aggregate law of motion $\hat{\varphi} : X \times Z \to X$ and allocations $\{(c_t, l_t^h, k_t^h)\}_{t\in\mathbb{N}}$, $\{(l_t^f, k_t^f)\}_{t\in\mathbb{N}}$ such that, for any stream $\{a_t\}_{t\in\mathbb{N}}$ of payoffs, we have

i) **Utility Maximization**
The representative household solves the following problem:

$$\begin{aligned} \max_{\{k_t^h\}_{t\in\mathbb{N}}} \quad & \sum_{t=0}^{\infty} \beta^t \cdot \mathbb{E}_0 \left[u \left[\mathcal{W}(\underline{k}_t, a_t) + (1 + \mathcal{R}(\underline{k}_t, a_t)) k_t^h - k_{t+1}^h \right] \right] \\ \text{subject to} \quad & k_{t+1}^h \in \left[0,\ \mathcal{W}(\underline{k}_t, a_t) + (1 + \mathcal{R}(\underline{k}_t, a_t)) k_t^h \right] \\ & \underline{k}_{t+1} = \hat{\varphi}(\underline{k}_t, a_t) \quad \text{for any } t \in \mathbb{N}. \end{aligned}$$

ii) **Profit Maximization**
For any $t \in \mathbb{N}$, the representative firm solves the following maximization problem:

$$\max_{k_t^f, l_t^f} \quad \left[F(k_t^f, l_t^f, a_t) - w_t l_t^f - (R_t + \delta) k_t^f \right].$$

iii) **Market Clearing and Consistency**
The market clears at any time t, that is,

$$\underline{k}_t = k_t^f = k_t^h \qquad \text{(Capital Market Clearing)}$$

$$\underline{l}_t = l_t^f = 1 \qquad \text{(Labor Market Clearing)}$$

$$y_t = f(\underline{k}_t, a_t) = c_t + \underline{k}_{t+1} - (1 - \delta)\underline{k}_t. \qquad \text{(Goods Market Clearing)}$$

Furthermore, the representative household's perceived aggregate law of motion ends up being the correct law of motion:

$$\varphi(\underline{k}, a) = \hat{\varphi}(\underline{k}, a)$$

for any $(\underline{k}, a) \in X \times Z$.

6.2.4 To the Recursive Representation

To obtain a recursive representation of the household's problem, note that in our specification above the household's sequence problem depends on two state variables: the individual state k_t and the aggregate state $\underline{\mathrm{k}}_t$. In practice, it is mathematically more tractable to treat both the aggregate state and the productivity shock as a pair of exogenous shocks. Heuristically, this is because the information on the aggregate states and the productivity shocks up to time t is equivalent to the information on the productivity shocks up to time t plus the initial aggregate state, since the aggregate state evolves deterministically according to the perceived law of motion $\hat{\varphi}$.

On the mathematical front, treating the aggregate state and the productivity shock as an extended exogenous shock in our recursive setting is made possible because $\{(\underline{\mathrm{k}}_t, a_t)\}_{t\in\mathbb{N}}$ is a Markov chain. Intuitively, this is a Markov chain because $\underline{\mathrm{k}}_{t+1}$ is determined by $\underline{\mathrm{k}}_t$ and a_t, while the distribution of a_{t+1} depends only on a_t. Formally, define $(S,\mathcal{S}) = (X,\mathcal{X})\bigotimes(Z,\mathcal{Z})$ as

$$\mathcal{P}((K,a),B) = \int_{X\times Z} I_B(\hat{\varphi}(K,a),a')P(a,da')$$

for any $B\in\mathcal{S}$ and $(K,a)\in S$. We know from the result on policy processes that $\mathcal{P}$ is a well-defined Markov kernel on $(S,\mathcal{S})$ such that

$$\int_S f(s')\mathcal{P}((K,a),ds') = \int_Z f(\hat{\varphi}(K,a),a')P(a,da')$$

for any $(K,a)\in S$ and $f\in\mathcal{S}_{+,b}$.

Now we are ready to reformulate the household's problem given this setting of an extended exogenous shock. Define the instantaneous return function $U: X\times X\times X\times Z\to\mathbb{R}$ and the constraint correspondence $\Gamma: X\times X\times Z\to X$ as

$$U(k,k',K,a) = u\left[\mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k-k'\right]$$

$$\Gamma(k,K,a) = [0,\ \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k]$$

for any $k,k'\in X$ and $(K,a)\in X\times Z$. It follows that a feasible plan started from the

initial values $(k_0, K_0, a_0) \in \mathbb{X} \times Z$ is defined as a sequence of functions $\{\pi_t\}_{t\in\mathbb{N}}$ such that

$$\pi_0 \in \Gamma(k_0, K_0, a_0)$$

$$\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t) \quad \text{for any } t \in \mathbb{N}, z^t \in (X \times Z)^t.$$

Denote the set of all feasible plans started from (k_0, K_0, a_0) as $\Pi(k_0, K_0, a_0)$.

The household's problem can now be written as

$$\max_{\pi\in\Pi(k_0,K_0,a_0)} \quad U(k_0, \pi_0, K_0, a_0) + \sum_{t=0}^{\infty} \beta^t \cdot \int_{(X\times Z)^t} U\left[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right] \mu^t((K_0, a_0), dz^t),$$

and the dual functional equation is

$$V(k,a;K) = \sup_{k'\in[0,\ \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k]} \left[u\left[\mathcal{W}(K,a) + (1+\mathcal{R}(K,a))k - k'\right] + \beta \cdot \int_Z V(k', z)\mathcal{P}((K,a), dz)\right]$$

$$= \sup_{k'\in[0,\ \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k]} \left[u\left[\mathcal{W}(K,a) + (1+\mathcal{R}(K,a))k - k'\right] + \beta \cdot \int_Z V(k', \hat{\varphi}(K,a), a')P(a, da')\right].$$

6.2.5 Recursive Competitive Equilibrium

An RCE in the stochastic neoclassical growth model is a collection of:

- A value function $V^* : X \times Z \times X \to \mathbb{R}$
- Policy functions $c^* : X \times Z \times X \to \mathbb{R}_+$ and $k^* : X \times Z \times X \to X$ of consumption and capital stock
- Price functions $\mathcal{W}, \mathcal{R} : X \times Z \to \mathbb{R}_+$
- The representative firm's capital demand function $K^f : Z \to X$
- A perceived law of motion $\hat{\varphi} : X \times Z \to X$ for the aggregate state variable

that satisfy the following conditions:

i) **Household Optimality**
The representative household optimizes given the perceived aggregate law of mo-

tion $\hat{\varphi}(\cdot)$ for the aggregate state. Specifically, V^* solves the functional equation

$$V(k,a;K) = \sup_{k' \in [0,\ \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k]} \Bigg[u\Big[\mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k-k'\Big] + \beta \cdot \int_Z V(k',a';\hat{\varphi}(K,a))P(a,da') \Bigg].$$

$k^* : X \times Z \times X \to X$ is the associated savings policy function, and $c^* : X \times Z \times X \to \mathbb{R}_+$ is defined as

$$c^*(k,a;K) = \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k-k^*(k,a;K)$$

for any $(k,a,K) \in X \times Z \times X$.

ii) **Profit Maximization**

For any level of capital and productivity $(K,a) \in X \times Z$, the first order conditions for profit maximization are satisfied:

$$\mathcal{W}(K,a) = F_l(K,1,a), \quad \mathcal{R}(K,a)+\delta = F_k(K,1,a).$$

iii) **Market Clearing and Belief Consistency**

The goods market clears given any level of capital and productivity $(K,a) \in X \times Z$:

$$c^*(K,a;K)+k^*(K,a;K)-(1-\delta)K = f(K,a).$$

In addition, the perceived aggregate law of motion for aggregate capital is the actual law of motion generated by the policy function:

$$K' = \hat{\varphi}(K,a) = k^*(K,a)$$

for any $(K,a) \in X \times Z$.

Note that we omit a capital market clearing condition because capital stock is predetermined by the representative household in the previous period. The role of the interest rate in each period is to ensure that the firm's demand for capital is precisely equal to the current capital stock owned by the household, which has already been determined in the previous period.

6.2.6 Solving for RCE

Now that we have defined RCE in this model, it remains to solve for the equilibrium. Checking the general assumptions in our case is easy, since we have to some extent already established the desired results when studying the social planner's decision making process. First, as is routine by now, we check whether the general assumptions hold.

A3 : **State Space and Space of Productivity Shocks**
The state space $X = [0, \bar{k}]$ is compact, as is the set $X \times Z$ on which the exogenous shocks take values. This indicates that $(X, \mathcal{X})$ and $(X, \mathcal{X}) \otimes (Z, \mathcal{Z})$ are both complete and separable subspaces of euclidean space.

A4 : **Feller Property of Markov Kernel $\mathcal{P}$**
We showed above that $\mathcal{P}$ has the Feller property, since P has the Feller property by assumption.

A5 : **Well-behaved Constraint Correspondence**
Define $h : X \times X \times Z \to \mathbb{R}_+$ as

$$h(k, K, a) = \mathcal{W}(K, a) + (1 + \mathcal{R}(K, a))k$$

for any $(k, K, a) \in X \times X \times Z$. By assumption, $\mathcal{W}$ and $\mathcal{R}$ are continuous functions on $X \times Z$, so h is a continuous function. The constraint correspondence can now be written as

$$\Gamma(k, K, a) = [0, h(k, K, a)],$$

so Γ is a non-empty compact valued and continuous correspondence.

A6 : **Well-behaved Instantaneous Return Function**

The return function $U : X \times X \times X \times Z \to \mathbb{R}$ is continuous on $X \times X \times X \times Z$, which is a compact set, so it is also bounded.

A7 : **Concavity Assumptions**

Fix the exogenous shock $(K,a) \in X \times Z$. Choose any $x, x' \in X \times X$ and $\lambda \in [0,1]$. Define

$$c = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))x_1 - x_2$$

$$c' = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))x_1' - x_2'$$

$$\tilde{c} = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))\left(\lambda x_1 + (1-\lambda)x_1'\right) - \left(\lambda x_2 + (1-\lambda)x_2'\right).$$

s Then,

$$\tilde{c} = \lambda c + (1-\lambda)c'$$

and therefore

$$\begin{aligned} \lambda \cdot U(x_1, x_2, K, a) + (1-\lambda) \cdot U(x_1', x_2', K, a) & \\ = \lambda u(c) + (1-\lambda)u(c') & \\ \leq u(\lambda c + (1-\lambda)c') = u(\tilde{c}) & \\ = U\left[\lambda x + (1-\lambda)x', K, a\right] & \end{aligned}$$

by the concavity of u, which shows us that $U(\cdot, K, a)$ is concave on $X \times X$.

Now fix $k \in X$ and $(K,a) \in X \times Z$. Choose any $\lambda \in (0,1)$ and distinct $x, x' \in X$. Again, define

$$c = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))k - x$$

$$c' = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))k - x'$$

$$\tilde{c} = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))k - \left(\lambda x + (1-\lambda)x'\right).$$

Then, c and c' are distinct values such that

$$\tilde{c} = \lambda c + (1-\lambda)c'.$$

It follows that

$$\begin{aligned}\lambda \cdot U(k,x,K,a) + (1-\lambda) \cdot U(k,x',K,a) & \\ &= \lambda u(c) + (1-\lambda)u(c') \\ &< u(\lambda c + (1-\lambda)c') = u(\tilde{c}) \\ &= U\left[k, \lambda x + (1-\lambda)x', K', a\right],\end{aligned}$$

by the strict concavity of u, which shows us that $U(k,\cdot,K,a)$ is a strictly concave function on X.

As for the constraint correspondence, first fix $(K,a) \in X \times Z$. Choose any $x, x' \in X \times X$, $y \in \Gamma(x,K,a)$ and $y' \in \Gamma(x',K,a)$, and $\lambda \in [0,1]$. Then, by definition,

$$0 \leq y \leq \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))x$$

$$0 \leq y' \leq \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))x',$$

so that

$$0 \leq \lambda y + (1-\lambda)y' \leq \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))(\lambda x + (1-\lambda)x').$$

This shows us that

$$\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x', K, a),$$

and as such that Γ is appropriately convex in the sense that we desire.

A9 : **Differentiability Assumptions**

For any $(K,a) \in X \times Z$, $U(\cdot,K,a)$ is twice continuously differentiable because u is twice continuously differentiable and its argument is linear in k and k'.

6.2.7 Existence of RCE

We want to establish first that an RCE does exist. Focusing first on the household's problem, it follows from the results we proved in the previous chapter that:

1) **Existence and Uniqueness of Solution to Functional Equation**
There exists a unique continuous and bounded solution V^* to the functional equation:

$$V^*(k,a;K) = \sup_{k' \in [0,\, \mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k]} \Bigg[u\Big[\mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k - k'\Big] + \beta \cdot \int_Z V^*(k',a';\hat{\varphi}(K,a))P(a,da') \Bigg],$$

for any $(k,a,K) \in X \times Z \times X$. Furthermore, $V^*(\cdot,a;K)$ is concave on X for any $(K,a) \in X \times Z$.

2) **Continuous and Single-valued Policy Correspondence**
There exists a continuous policy function $k^* : X \times Z \times X \to X$ such that

$$V^*(k,a;K) = u\Big[\mathcal{W}(K,a)+(1+\mathcal{R}(K,a))k - k^*(k,a;K)\Big] + \beta \cdot \int_Z V^*(k^*(k,a;K),a';\hat{\varphi}(K,a))P(a,da')$$

for any $(k,a,K) \in X \times Z \times X$. Similarly, there exists a continuous optimal consumption policy funciton $c^* : X \times Z \times X \to \mathbb{R}_+$ defined as

$$c^*(k,a;K) = \mathcal{W}(K,a) + (1 + \mathcal{R}(K,a))k - k^*(k,a;K)$$

for any $(k,a,K) \in X \times Z \times X$.

3) **Envelope Theorem for Value Function**
For any $(K,a) \in X \times Z$ and $k \in (0,\bar{k})$, assuming that $k^*(k,a;K) \in \Gamma(k,K,a)^o$, we have

$$\frac{\partial V^*(k,a;K)}{\partial k} = \frac{\partial U(k,k^*(k,a;K)K,a)}{\partial k} = u'\left[c^*(k,a;K)\right] \cdot (1 + \mathcal{R}(K,a)).$$

Assuming that $k^*(k,a;K) \in \Gamma(k,K,a)^o$ for any $(k,a,K) \in X \times Z \times X$, the first order condition for maximization tells us that

$$-u'\Big[c^*(k,a;K)\Big] + \beta \cdot \int_Z \frac{\partial V^*(c^*(k,a;K),a';\hat{\varphi}(K,a))}{\partial k} P(a,da') = 0.$$

Together with the envelope theorem for the value function, we have

$$u'\Big[c^*(k,a;K)\Big] = \beta \cdot \int_Z u'\Big[c^*\left(k^*(k,a;K),a';\hat{\varphi}(K,a)\right)\Big] \cdot (1 + \mathcal{R}(\hat{\varphi}(K,a),a)) P(a,da')$$

for any $(k,a,K) \in X \times Z \times X$.

Meanwhile, the profit maximiztion condition tells us that the equilibrium wage and interest rate functions are given as

$$\mathcal{W}(K,a) = F_l(K,1,a), \quad \mathcal{R}(K,a) + \delta = F_k(K,1,a)$$

for any $(K,a) \in X \times Z$. Finally, belief consistency condition implies that $\hat{\varphi}(\cdot) = g(\cdot)$; defining the aggregate policy functions as

$$C^*(K,a) = c^*(K,a;K), \quad K^*(K,a) = k^*(K,a;K)$$

for any $(K,a) \in X \times Z$, the Euler equation becomes

$$u'\Big[C^*(K,a)\Big] = \beta \cdot \int_Z u'\Big[C^*\left(K^*(K,a),a'\right)\Big] \cdot \left(F_k(K^*(K,a),1,a) + (1-\delta)\right) P(a,da')$$

for any $(K,a) \in X \times Z$, which is precisely the Euler equation corresponding to the social planner's problem.

The existence of RCE now follows if we can just show that the goods market equilibrium holds, but this follows immediately from the representative household's budget constraint: for any $(K,a) \in X \times Z$,

$$c^*(K,a;K) + k^*(K,a;K) = \mathcal{W}(K,a) + (1+\mathcal{R}(K,a))K = F_l(K,1,a) + (1-\delta)\,K + F_k(K,1,a) \cdot K,$$

and since

$$F_l(K,1,a) + F_k(K,1,a)K = F(K,1,a)$$

by the CRS property of the production function $F(\cdot,a)$, we have

$$c^*(K,a;K) + k^*(K,a;K) - (1-\delta)K = F(K,1,a).$$

Chapter 7

Models of Heterogeneous Households

So far, we have been assuming a representative agent framework, where homogenoeus households and firms make the same consumption, saving and production decisions. A natural extension of this basic framework is to induce household heterogeneity by introducing a component such as individual labor productivity that differs across households. This extension allows us to obtain not only the dynamics of aggregate macroeconomic variables, but also investigate how wealth is distributed across households in the economy.

7.1 The Income Fluctuation Problem

We first study the income fluctuation problem, which concerns a single household subject to endowment shocks and are price takers. The model allows us to study how price taking households make consumption and savings decisions in the presence of incomplete asset markets. Like in the stochastic neoclassical growth model, it turns out that there exists some stationary distribution to which household savings converge.

7.1.1 Model Assumptions

The assumptions of the basic consumption-savings model are as follows:

Endowment Each period, the household receives an endowment y_t that is generated by a

Markov kernel P on the compact space $(Y, \mathcal{Y})$, where Y is given as the closed interval

$$Y = [\underline{y}, \overline{y}].$$

We require $\underline{y} > 0$, so that, even in the worst case, the household receives a positive endowment. We assume that P satisfies the Feller propery and is monotonic.

Savings The household can save in a single (non-contingent) asset each period. As such, consumption is not fully insured in this model. The rate of return to savings is governed by a fixed interest rate $r > 0$, and the household is subject to a borrowing constraint $a_t > -\phi$, where $\phi \geq 0$.

When $r > 0$, the natural borrowing limit (NBL) is the maximum amount that the household can borrow at any time that it will be able to pay back no matter what, even when it faces the worst realizations of the endowment and does not consume forever. The NBL $-b$ must therefore be given as

$$-b = -\sum_{j=1}^{\infty} \left(\frac{\underline{y}}{1+r} \right)^j = -\frac{\underline{y}}{r}.$$

We assume that our borrowing limit satisfies

$$r\phi - \underline{y} < 0;$$

when $r > 0$, this condition can be understood as requiring the borrowing limit to be more stringent than the NBL.

Savings takes values in the space

$$X = [-\phi, +\infty),$$

with $\mathcal{X}$ as the corresponding Borel σ-algebra.

Preferences The representative consumer's flow utility function is $u : \mathbb{R}_+ \to \mathbb{R}$. u is assumed to be twice continuously differentiable on $(0, +\infty)$ with derivatives

$$u'(\cdot) > 0, \quad u''(\cdot) < 0$$

on $\mathbb{R}$, so that u is strictly concave on $\mathbb{R}$. We assume in addition that the Inada conditions hold:

$$\lim_{c \downarrow 0} u'(c) = +\infty, \quad \lim_{c \to \infty} u'(c) = 0.$$

Finally, we assume that there exists an upper bound on the coefficient for relative risk aversion: there exists a $\mu > 0$ such that

$$0 < -c\frac{u''(c)}{u'(c)} \leq \mu$$

for any $c > 0$.

For computational purposes, we also make the (strong) assumption that u is a bounded function. This assumption is actually violated in many cases by some frequently used utility functions (such as log or CRRA utility) but is satisfied by utility functions that have an exponential form.

The household discounts future utility by the discount factor $\beta \in (0, 1)$.

Under the above assumptions, the flow budget constraint of the household at time t is given as

$$c_t + a_{t+1} = y_t + (1 + r)a_t,$$

where y_t is the time t realization of endowment and a_t is the household's wealth at time t. The household's sequence problem is then given as

$$\begin{aligned} \max_{\{c_t\}_{t \in \mathbb{N}}} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \right] \\ \text{subject to} \quad & a_t \geq -\phi \quad c_t \geq 0 \\ & c_t + a_{t+1} = (1 + r)a_t + y_t, \quad \text{for any } t \in \mathbb{N}, \end{aligned}$$

with state-only formulation

$$\max_{\{a_t\}_{t\in\mathbb{N}}} \quad \sum_{t=0}^{\infty} \beta^t \cdot \mathbb{E}_0 \left[u\left[(1+r)a_t + y_t - a_{t+1} \right] \right]$$

$$\text{subject to} \quad a_{t+1} \in [-\phi,\ (1+r)a_t + y_t] \quad \text{for any } t \in \mathbb{N}.$$

The constraint correspondence $\Gamma : X \times Y \to X$ of this problem is defined as

$$\Gamma(a, y) = [-\phi,\ y + (1+r)a]$$

for any $(a, y) \in X \times Y$. The constraint correspondence is non-empty for any $(a, y) \in X \times Y$ because

$$y + (1+r)a \geq \underline{y} - (1+r)\phi > -\phi$$

by the assumption that $r\phi < \underline{y}$, so that $-\phi \in [-\phi,\ y + (1+r)a]$.

Therefore, the functional equation corresponding to this problem is

$$V(a, y) = \sup_{a' \in \Gamma(a,y)} \left[u[(1+r)a + y - a'] + \beta \cdot \int_Y V(a', y') P(y, dy') \right],$$

or equivalently,

$$V(a, y) = \max_{(c,a')} \left[u(c) + \beta \cdot \int_Y V(a', y') P(y, dy') \right]$$

$$\text{subject to} \quad c + a' = y + (1+r)a$$

$$a' \geq -\phi.$$

7.1.2 Solving for the Optimal Policy Functions

We can show that the six general assumptions are satisfied under our model assumptions:

B1 : **State Space and Space of Stochastic Shocks**

The state space X and space of stochastic shocks Y are subset of $\mathbb{R}$, so that they

are complete and separable metric spaces equipped with the euclidean metric.

B2 : **Feller Property of Markov Kernel** P

P has the Feller property by assumption.

B3 : **Well-behaved Constraint Correspondence**

For any $(a, y) \in X \times Y$, $\Gamma(a, y)$ is a non-empty closed interval on $\mathbb{R}$, where the upper endpoint is a continuous function of (a, y). Therefore, it is a non-empty compact, convex valued and continuous correpsondence.

B4 : **Well-behaved Instantaneous Return Function**

The return function $U : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ defined as

$$U(a, a', y) = u\left[(1+r)a + y - a'\right],$$

where $\mathrm{Gr}(\Gamma) = \{(a, a', y) \mid a' \in \Gamma(a, y)\}$ is the graph of $\Gamma(\cdot)$, is a continuous function and bounded by assumption.

B5 : **Monotonicity Assumptions**

For any $(a', y) \in X \times Y$, the function $U(\cdot, a', y)$ is a strictly increasing function, and

$$\Gamma(a_1, y) = [-\phi, (1+r)a_1 + y] \cap X \subset [-\phi, (1+r)a_2 + y] \cap X = \Gamma(a_2, y)$$

for any $y \in Y$ and $a_1, a_2 \in X$ such that $a_1 < a_2$.

Similarly, for any $(a, a') \in X \times X'$, the function $U(a, a', \cdot)$ is strictly increasing, and

$$\Gamma(a, y_1) = [-\phi, (1+r)a + y_1] \cap X \subset [-\phi, (1+r)a + y_2] \cap X = \Gamma(a, y_2)$$

for any $a \in X$ and $y_1, y_2 \in Y$ such that $y_1 < y_2$. Furthermore, P is a monotonic kernel.

B6 : **Concavity Assumptions**

Choose any $\lambda \in (0,1)$ and $(a,y) \in X \times Y$ and distinct $(a', a'') \in X$; then, because $u(\cdot)$ is a strictly concave function and

$$y + (1+r)a - \left(\lambda a' + (1-\lambda)a''\right) = \lambda\left(y + (1+r)a - a'\right) + (1-\lambda)\left(y + (1+r)a - a''\right),$$

where $y + (1+r)a - a' \neq y + (1+r)a - a''$, we have

$$\begin{aligned} U(a, \lambda a' + (1-\lambda)a'', y) &> \lambda \cdot u\left[y + (1+r)a - a'\right] + (1-\lambda)u\left[y + (1+r)a - a''\right] \\ &= \lambda U(a, a', y) + (1-\lambda)U(a, a'', y). \end{aligned}$$

This proves that the mapping

$$a' \mapsto U(a, a', \epsilon)$$

is strictly concave on X.

Now choose any $y \in Y$ and pairs $(a_1, a_1'), (a_2, a_2') \in X \times X$. Then, for any $\lambda \in [0,1]$,

$$\begin{aligned} y + (1+r)\left(\lambda a_1 + (1-\lambda)a_2\right) - \left(\lambda a_1' + (1-\lambda)a_2'\right) \\ = \lambda\left[y + (1+r)a_1 - a_1'\right] + (1-\lambda)\left[y + (1+r)a_2 - a_2'\right], \end{aligned}$$

and the concavity of $u(\cdot)$ implies that

$$U\left[\lambda(a_1, a_1') + (1-\lambda)(a_2, a_2'), y\right] \geq \lambda U(a_1, a_1', y) + (1-\lambda)U(a_2, a_2', y).$$

Therefore, $U(\cdot, y)$ is concave on $X \times X$.

To show the convexity of the constraint correspondence, choose any $y \in Y$ and

$a, a' \in X$. Then, for any $\lambda \in [0,1]$ and $x \in \Gamma(a,y)$, $x' \in \Gamma(a',y)$, we have

$$-\phi \leq x \leq y + (1+r)a$$

$$-\phi \leq x' \leq y + (1+r)a',$$

so that

$$-\phi \leq \lambda x + (1-\lambda)x' \leq y + (1+r)\left(\lambda a + (1-\lambda)a'\right).$$

Since $\lambda x + (1-\lambda)x' \in X$ by the convexity of X, this implies that

$$\lambda x + (1-\lambda)x' \in \Gamma(\lambda a + (1-\lambda)a', y),$$

so that $\Gamma(\cdot)$ is appropriately convex.

B7 : **Differentiability Assumptions**

X has a non-empty interior by assumption, and for any $y \in Y$, the mapping

$$(a,a') \mapsto U(a,a',y) = u\left[y + (1+r)a - a'\right]$$

is clearly continuously differentaible on the interior of $\mathrm{Gr}(\Gamma; y)$.

We can immediately conclude that the following general results hold:

1) **Existence and Uniqueness of Solution to Functional Equation**

There exists a unique continuous and bounded solution V^* to the functional equation:

$$V(a,y) = \sup_{a' \in \Gamma(a,y)} \left[u[(1+r)a + y - a'] + \beta \cdot \int_Y V(a',y')P(y',dy)\right],$$

for any $(a,y) \in A \times y$. In addition, $V^*(\cdot, y)$ is strictly concave on X for any $y \in Y$, and $V^*(\cdot)$ is strictly increasing in both arguments.

2) **Continuous and Single-valued Policy Correspondence**
There exists a continuous policy function $a^* : X \times Y \to X$ such that

$$V^*(a,y) = u[(1+r)a + y - a^*(a,y)] + \beta \cdot \int_Y V^*(a^*(a,y),y')P(y',dy),$$

for any $(a,y) \in X \times Y$. We define the consumption policy function $c^* : X \times Y \to \mathbb{R}_+$ as

$$c^*(a,y) = (1+r)a + y - a^*(a,y)$$

for any $(a,y) \in A \times Y$.

3) **Envelope Theorem for Value Function**
For any $(a,y) \in X^o \times Y$ such that $a^*(a,y) \in \Gamma(a,y)^o$, $V^*(\cdot,y)$ is differentiable at a with derivative

$$\nabla_a V^*(a,y) = \nabla_a U(a,a^*(a,y),y) = (1+r) \cdot u'[c^*(a,y)].$$

7.1.3 The Euler Inequality

To obtain a succinct characterization of the dynamic properties of the model in the form of the Euler equation, we want to make the envelope theorem more widely applicable by relaxing the requirement that the envelope theorem holds only for those (a,y) such that $a^*(a,y) \in \Gamma(a,y)^o$. It turns out that the unique structure of this problem allows us to entirely drop that assumption.

Lemma (Implications of Positive Consumption)
Consider the consumption-savings model above. Then, the following hold true:

i) For any $(a,y) \in X \times Y$, we have $c^*(a,y) > 0$.

ii) For any $y \in Y$, $V^*(\cdot, y)$ is differentiable on $(-\phi, +\infty)$ with derivative given as

$$\nabla_a V^*(a, y) = (1+r) \cdot u'[c^*(a, y)]$$

for any $a \in (-\phi, +\infty)$.

iii) For any $(a, y) \in X \times Y$ such that $a^*(a, y) > -\phi$, we have

$$\nabla_a V^*(a^*(a, y), y') = (1+r) \cdot u'\left[c^*(a^*(a, y), y')\right].$$

Proof) i) Choose any $(a, y) \in [-\phi, +\infty) \times Y$ and suppose that $c^*(a, y) = 0$. This implies that

$$a^*(a, y) = (1+r)a + y \geq -\phi.$$

If this inequality holds as an equality, then it must be the case that

$$-\phi = y + (1+r)a \geq \underline{y} - (1+r)\phi \geq -\phi,$$

so that we have

$$\underline{y} - (1+r)\phi = -\phi,$$

or $r\phi = \underline{y}$, which contradicts the fact that we chose ϕ to satify $r\phi < \underline{y}$.

Therefore, it must be the case that $a^*(a, y) > -\phi$. Here, we can derive a contradiction by showing that the concavity of the value function and the Inada condition for the utility function can allow us to increase lifetime utility by slightly lowering savings and slightly increasing consumption compared to $a^*(a, y)$ and $c^*(a, y)$, which contradicts their optimality.

Formally, note that, because V^* is a bounded function, $\{V^*(\cdot, y')\}_{y' \in Y}$ is a uniformly bounded collection of concave functions on X. It follows from the equi-Lipschitz property of convex functions (which are basically

concave functions times -1) that there exists an $L > 0$ such that

$$\left|V^*(a_1, y') - V^*(a_2, y')\right| < L|a_1 - a_2|$$

for any $y' \in Y$ and $a_1, a_2 \in [-\phi, a^*(a, y)]$.
Meanwhile, by the Inada condition

$$\lim_{c \downarrow 0} u'(c) = +\infty,$$

there exists some $\epsilon \in (0, a^*(a, y) - \phi)$ such that $u'(c) > L$ for any $0 < c \leq \epsilon$. From the mean value theorem, we can tell that there exists some $\tilde{\epsilon} \in (0, \epsilon)$ such that

$$u(0) - u(\epsilon) = -u'(\tilde{\epsilon})\epsilon.$$

Since $\tilde{\epsilon} < \epsilon$, it follows that $u'(\tilde{\epsilon}) > L$, so that

$$u(0) - u(\epsilon) < -L\epsilon.$$

Now, defining $\mathbf{a} = a^*(a, y) - \epsilon$, we have

$$-\phi < \mathbf{a} < a^*(a, y) = y + (1 + r)a,$$

so that $\mathbf{a} \in \Gamma(a, y)$, and by the equi-Lipschitz property we derived above,

$$\left|V^*(a^*(a, y), y') - V^*(\mathbf{a}, y')\right| < L\epsilon$$

for any $y' \in Y$ and therefore

$$\beta \int_Y \left(V^*(a^*(a, y), y') - V^*(\mathbf{a}, y')\right) P(y, dy') \leq \beta L\epsilon.$$

Putting all this together, we can see that

$$V^*(a, y) - \left(u((1 + r)a + y - \mathbf{a}) + \beta \int_Y V^*(\mathbf{a}, y') P(y, dy')\right)$$

$$= u(0) - u(\epsilon) + \beta \int_Y \left(V^*(a^*(a, y), y') - V^*(\mathbf{a}, y')\right) P(y, dy')$$

$$< -L\epsilon + \beta L\epsilon = (1-\beta)L\epsilon < 0.$$

This contradicts the fact that

$$V^*(a,y) = \sup_{a' \in \Gamma(a,y)} \left[u((1+r)a + y - a') + \beta \int_Y V^*(a',y') P(y,dy') \right],$$

so we must have $c^*(a,y) > 0$.

ii) The positivity of consumption indicates that, for any $(a,y) \in X \times Y$, we have

$$y + (1+r)a - a^*(a,y) = c^*(a,y) > 0,$$

or in other words that

$$a^*(a,y) \in [-\phi,\ y + (1+r)a).$$

Recall from the proof of the envelope theorem that $V^*(\cdot, y)$ is differentiable at some $a_0 \in X$ if there exists an $\epsilon > 0$ such that $a^*(a_0, y) \in \Gamma(a,y)$ for any $a \in B(a_0, \epsilon)$. We will show that this is automatically satisfied for any $a_0 > \underline{a}$ and $y \in Y$.

Since a_0 is contained in the interior of X, by definition there exists some $\eta > 0$ such that $B(a_0, \eta)$. Furthermore, $a^*(a_0, y) \in [-\phi,\ y + (1+r)a_0)$, so that we can define

$$\epsilon = \min \left(\frac{y + (1+r)a_0 - a^*(a_0,y)}{1+r},\ \eta \right) > 0.$$

Then, for any $a \in B(a_0, \epsilon)$, we can see that $a \in X$ and, by our choice of $\epsilon > 0$, since $(1+r)a_0 - (1+r)\epsilon < (1+r)a$,

$$y + (1+r)a > y + (1+r)a_0 - (y + (1+r)a_0 - a^*(a_0,y)) = a^*(a_0,y).$$

Since $a^*(a_0,y) \geq -\phi$ and $a^*(a_0,y) \in X$ by the definition of $\Gamma(\cdot)$, it follows

that

$$a^*(a_0, y) \in \Gamma(a, y) = [-\phi,\ y + (1+r)a].$$

We can conclude that, for any $a \in X^o = (-\phi, +\infty)$, $V^*(\cdot, y)$ is differentiable at a with derivative given as

$$\nabla_a V^*(a, y) = u'[c^*(a, y)](1+r).$$

iii) Choose any $(a, y) \in X \times Y$ such that $a^*(a, y) > -\phi$. Applying the preceding result immediately shows us that

$$\nabla_a V^*(a^*(a, y), y') = (1+r) \cdot u'\left[c^*(a^*(a, y), y')\right]$$

for any $y' \in Y$.

Q.E.D.

Lemma (Right Directional Derivative of the Value Function)
Consider the consumption-savings model above. For any $y \in Y$, the right directional derivative of $V^*(\cdot, y)$ at $-\phi$ exists, that is, there exists some $\nabla_{a+} V^*(-\phi, y) \in (-\infty, +\infty]$ such that

$$\lim_{h \downarrow 0} \frac{V^*(-\phi + h, y) - V^*(-\phi, y)}{h} = \nabla_{a+} V^*(-\phi, y).$$

Furthermore, this derivative satisfies the inequality

$$\nabla_{a+} V^*(-\phi, y) \geq u'[c^*(-\phi, y)](1+r).$$

Proof) We first establish that the right directional derivative exists. For any $y \in Y$,

$V^*(\cdot, y)$ is a strictly concave function. Choose any $0 < h_1 < h_2$, and define $t = \frac{h_1}{h_2} \in (0,1)$. By concavity,

$$V^*((1-t)(-\phi)+t(-\phi+h_2), y) > (1-t)V^*(-\phi, y)+tV^*(-\phi+h_2, y),$$

where $(1-t)(-\phi)+t(-\phi+h_2) = -\phi+h_1$. Therefore,

$$\begin{aligned} V^*(-\phi+h_1, y)-V^*(-\phi, y) &> t\,[V^*(-\phi+h_2, y)-V^*(-\phi, y)] \\ &= \frac{h_1}{h_2}\,[V^*(-\phi+h_2, y)-V^*(-\phi, y)], \end{aligned}$$

rearranging which we have

$$\frac{V^*(-\phi+h_1, y)-V^*(-\phi, y)}{h_1} > \frac{V^*(-\phi+h_2, y)-V^*(-\phi, y)}{h_2} > 0,$$

where the last inequality follows from the fact that $V^*(\cdot, y)$ is strictly increasing on X. $\left\{\frac{V^*(-\phi+h, y)-V^*(-\phi, y)}{h}\right\}_{h>0}$ is thus a sequence of non-negative reals that are strictly decreasing with h, so that it converges to some $\nabla_{a+} V^*(-\phi, y) \in (-\infty, +\infty]$ as $h \downarrow 0$.

To see the inequality, note that

$$a^*(-\phi, y) \in [-\phi, y-(1+r)\phi] \subset [-\phi, y+(1+r)a] = \Gamma(a, y)$$

for any $a \in [-\phi, +\infty)$. In other words,

$$V^*(a, y) \geq u\,[y+(1+r)a-a^*(-\phi, y)]+\beta \int_Y V^*(a^*(-\phi, y), y_0)P(y, dy')$$

for any $a \geq -\phi$, with the inequality holding as an equality if $a = -\phi$. Thus, for any $h > 0$,

$$\begin{aligned} \frac{V^*(-\phi+h, y)-V^*(-\phi, y)}{h} &\geq \frac{u\,[y+(1+r)(-\phi+h)-a^*(-\phi, y)]-u\,[c^*(-\phi, y)]}{h} \\ &= \frac{u\,[c^*(-\phi, y)+(1+r)h]-u\,[c^*(-\phi, y)]}{(1+r)h}(1+r). \end{aligned}$$

Taking $h \downarrow 0$ on both sides shows us that

$$\nabla_{a+} V^*(-\phi, y) \geq u'[c^*(-\phi, y)](1+r).$$

Q.E.D.

We can now establish the Euler equation in a straightforward manner. For any $(a, y) \in X \times Y$, we consider two cases. If $a^*(a, y) > -\phi$, that is, if the borrowing constraint does not bind given wealth a, then the first order condition for maximization can be applied to see that

$$-u'[c^*(a, y)] + \beta \int_E \nabla_a V^*(a^*(a, y), y') P(y, dy') = 0.$$

Now, the envelope theorem yields the Euler equation

$$u'[c^*(a, y)] = \beta(1+r) \int_Y u' \left[c^*(a^*(a, y), y') \right] P(y, dy').$$

On the other hand, suppose $a^*(a, y) = -\phi$. By the definition of the maximizer we can see that

$$u\left[y + (1+r)a - a'\right] + \beta \int_Y V^*(a', y') P(y, dy')$$

$$< u[y + (1+r)a + \phi] + \beta \int_Y V^*(-\phi, y') P(y, dy')$$

for any $a' > -\phi$. By implication, for any $h > 0$,

$$\frac{u'[c^*(a, y) - h] - u'[c^*(a, y)]}{h} + \beta \int_Y \frac{V^*(-\phi + h, y') - V^*(-\phi, y')}{h} P(y, dy') < 0,$$

or

$$\frac{u'[c^*(a, y) - h] - u'[c^*(a, y)]}{-h} > \beta \int_Y \frac{V^*(-\phi + h, y') - V^*(-\phi, y')}{h} P(y, dy').$$

Taking $h \downarrow 0$ yields, by the MCT,

$$u'[c^*(a,y)] \geq \beta \int_Y \nabla_{a+} V^*(-\phi, y') P(y, dy').$$

Furthermore, since

$$\nabla_{a+} V^*(-\phi, y') \geq u'\left[c^*(-\phi, y')\right](1+r)$$

for any $y' \in Y$, the monotonicity of integration shows us that

$$u'[c^*(a,y)] \geq \beta(1+r) \int_Y u'\left[c^*(-\phi, y')\right] P(y, dy').$$

In any case, the following Euler inequality holds true:

$$u'[c^*(a,y)] \geq \beta(1+r) \int_E u'\left[c^*(a^*(a,y), y')\right] P(y, dy'),$$

where the inequality holds as an equality if $a^*(a,y) > -\phi$.

7.1.4 Montonicity of Consumption and Savings

Having established the Euler inequality, we can now show that the policy functions are strictly monotonic in current wealth as an implication of the envelope theorem and the Euler inequality.

Theorem (Monotonicity of Policy Functions in Wealth)
Consider the consumption-savings model above. For any $y \in Y$, $c^*(\cdot, y)$ is strictly increasing on X, and $a^*(\cdot, y)$ is increasing on X.

Proof) We first establish that $c^*(\cdot, y)$ is increasing on X. Choose any $y \in Y$ and $a_1, a_2 \in X$ such that $-\phi \leq a_1 < a_2$. The fact that the borrowing constraint might be binding is a slight problem, but we can easily circumvent it by instead considering a sequence $\{a_1^n\}_{n \in N_+}$ such that $a_1 < a_1^n < a_2$ and $a_1^n \searrow a_1$ as $n \to \infty$.

For each $n \in N_+$, since $a_1^n < a_2$ and $a_1^n, a_2 \in (-\phi, +\infty) \subset X$, we can apply the envelope theorem to see that

$$\nabla_a V^*(a_1^n, y) = u'[c^*(a_1^n, y)](1+r), \quad \nabla_a V^*(a_2, y) = u'[c^*(a_2, y)](1+r).$$

Since $V^*(\cdot, y)$ is a strictly concave function, it follows that $\nabla_a V^*(\cdot, y)$ is a strictly decreasing function on X^o, so that

$$u'[c^*(a_1^n, y)](1+r) > u'[c^*(a_2, y)](1+r)$$

and therefore

$$c^*(a_1^n, y) < c^*(a_2, y).$$

Applying the same line of reasoning to $a_1^{n+1} < a_1^n$ reveals that

$$c^*(a_1^n, y) \leq c^*(a_1^1, y) < c^*(a_2, y)$$

for any $n \in N_+$. Now the continuity of $c^*(\cdot, y)$ on X and the fact that a_1^n decreases to a_1 as $n \to \infty$ implies that

$$c^*(a_1, y) < c^*(a_2, y).$$

To connect this result to strict monotonicity for the savings policy function in wealth, choose any $y \in Y$, $a_1, a_2 \in X$ such that $a_1 < a_2$, and suppose for the sake of contradiction that $a^*(a_1, y) > a^*(a_2, y)$.This implies that $a_1^*(a_1, y) > -\phi$, and therefore the Euler equation holds:

$$u'\left[c^*(a_1, y)\right] = \beta(1+r)\int_E u'\left[c^*(a^*(a_1, y), y')\right] P(y, dy').$$

Meanwhile, the Euler inequality holds for $a^*(a_2, y)$:

$$u'\left[c^*(a_2, y)\right] \geq \beta(1+r)\int_E u'\left[c^*(a^*(a_2, y), y')\right] P(y, dy').$$

The monotonicity of $c^*(\cdot, y)$ on X that we showed above now implies that

$$u'\left[c^*(a^*(a_1, y), y')\right] \leq u'\left[c^*(a^*(a_2, y), y')\right]$$

for any $y' \in Y$, so that

$$\begin{aligned} u'\left[c^*(a_1, y)\right] &= \beta(1+r)\int_E u'\left[c^*(a^*(a_1, y), y')\right] P(y, dy') \\ &\leq \beta(1+r)\int_E u'\left[c^*(a^*(a_2, y), y')\right] P(y, dy') \leq u'\left[c^*(a_2, y)\right]. \end{aligned}$$

This implies

$$c^*(a_1, y) \geq c^*(a_2, y),$$

which contradicts the fact that we must have $c^*(a_1, y) < c^*(a_2, y)$ becauase $c^*(\cdot, y)$ is strictly monotonic and $a_1 < a_2$. Therefore, we must have $a^*(a_1, y) \leq a^*(a_2, y)$, which establishes the monotonicity of $a^*(\cdot, y)$.

Q.E.D.

The above result has an immediate application. For any $y \in Y$ and $a > -\phi$, $c^*(-\phi, y) < c^*(a, y)$ implies

$$u'\left[c^*(-\phi, y)\right] > u'\left[c^*(a, y)\right].$$

Since $a \in (-\phi, +\infty) = X^o$, the envelope theorem implies that

$$\nabla_a V^*(a,y) = u'[c^*(a,y)](1+r),$$

while we saw in one of the preceding lemmas that

$$\nabla_{a+} V^*(-\phi,y) \geq u'[c^*(-\phi,y)](1+r).$$

Therefore, we are able to conclude that

$$\nabla_{a+} V^*(-\phi,y) > \nabla_a V^*(a,y).$$

Note also that we can now see that $\nabla_{a+} V^*(-\phi,y)$ is real-valued. For any $h > 0$, the mean value theorem tells us that

$$\frac{V^*(-\phi+h,y) - V^*(-\phi,y)}{h} = \nabla_a V^*(\tilde{a},y)$$

for some $\tilde{a} \in (-\phi, -\phi+h)$. Applying the envelope theorem reveals that

$$\frac{V^*(-\phi+h,y) - V^*(-\phi,y)}{h} = u'[c^*(\tilde{a},y)](1+r) < u'[c^*(-\phi,y)](1+r),$$

since $c^*(\tilde{a},y) > c^*(-\phi,y)$ by strict monotonicity of consumption in wealth. This holds for any $h > 0$, so

$$\nabla_a V^*(a,y) < \nabla_{a+} V^*(-\phi,y) \leq u'[c^*(-\phi,y)](1+r).$$

Defining $\nabla_a V^*(-\phi,y) = \nabla_{a+} V^*(-\phi,y)$, we can treat $\nabla_a V^*(\cdot,y)$ as a strictly decreasing function on X. Note also that $\nabla_a V^*(-\phi,\cdot)$ is a non-negative measurable function, so that its integral with respect to any measure on $(Y,\mathcal{Y})$ is well-defined. This will come in handy in what comes next.

Unlike monotonicity in wealth, the monotonicity of consumption and savings in endowments does not follow immediately because the dependence of tomorrow's endowment on today's endowment complicates how the household responds to changes in income shocks. The following furnishes independence of the shocks as a sufficient condition for optimal savings and consumption to also be monotonic in y.

Theorem (Monotonicity of Policy Functions in Endowment)
Consider the consumption-savings model above, and suppose that the endowment shocks are iid, that is, there exists a distribution λ on $(Y, \mathcal{Y})$ such that $P(y, \cdot) = \lambda$ for any $y \in Y$. Then, for any $a \in X$, $a^*(a, \cdot)$ is an increasing function on Y and $c^*(a, \cdot)$ is strictly increasing on Y.

Proof) **Monotonicity of Savings Policy Function**

The proof hinges on the following construction. Choose any $a \in X$ and $y_1, y_2 \in Y$ such that $y_1 < y_2$, and suppose that $a^*(a, y_1) > a^*(a, y_2)$. Then,

$$c^*(a, y_1) = y_1 + (1+r)a - a^*(a, y_1) < y_2 + (1+r)a - a^*(a, y_2) = c^*(a, y_2),$$

so that

$$u'[c^*(a, y_1)] > u'[c^*(a, y_2)].$$

Meanwhile, since $a^*(a, y_1) > -\phi$, the Euler equation holds as

$$u'[c^*(a, y_1)] = \beta \int_E \nabla_a V^*(a^*(a, y_1), y') P(y_1, dy'),$$

and the Euler inequality holds for $a^*(a, y_2)$:

$$u'[c^*(a, y_2)] \geq \beta \int_E \nabla_a V^*(a^*(a, y_2), y') P(y_2, dy').$$

We will be able to derive a contradiction if

$$\int_E \nabla_a V^*(a^*(a, y_1), y') P(y_1, dy') \leq \int_E \nabla_a V^*(a^*(a, y_2), y') P(y_2, dy'),$$

Suppose now that $\lambda = P(y, \cdot)$ for any $y \in Y$. Then, because $\nabla_a V^*(\cdot, y')$ is a strictly decreasing function on X for each $y' \in Y$, we have

$$\nabla_a V^*(a^*(a, y_1), y') \leq \nabla_a V^*(a^*(a, y_2), y')$$

for any $y' \in Y$. It follows that

$$\int_E \nabla_a V^*(a^*(a, y_1), y') d\lambda(y') \leq \int_E \nabla_a V^*(a^*(a, y_2), y') d\lambda(y'),$$

and as such we have

$$u'\left[c^*(a,y_1)\right] \leq u'\left[c^*(a,y_2)\right],$$

a contradiction. Thus, it must be the case that $a^*(a,y_1) \leq a^*(a,y_2)$.

Monotonicity of Consumption Policy Function

Choose any $a \in X$ and $y_1, y_2 \in Y$ such that $y_1 < y_2$. Suppose $a^*(a,y_2) > -\phi$, so that the Euler inequality holds as an equality at (a,y_2). We already showed that, under the above conditions, we have $a^*(a,y_1) \leq a^*(a,y_2)$. We now examine two cases.

First, suppose $a^*(a,y_1) = a^*(a,y_2)$. Then,

$$\begin{aligned} c^*(a,y_1) &= y_1 + (1+r)a - a^*(a,y_1) \\ &= y_1 + (1+r)a - a^*(a,y_2) \\ &< y_2 + (1+r)a - a^*(a,y_2) = c^*(a,y_2), \end{aligned}$$

so that we immediately obtain our desired result.

Next, suppose $a^*(a,y_2) = -\phi$. Then, from the monotonicity result earlier we can see that $a^*(a,y_1) = -\phi$ as well, so that $c^*(a,y_1) < c^*(a,y_2)$ again.

Finally, suppose that $a^*(a,y_1) < a^*(a,y_2)$ and $a^*(a,y_2) > -\phi$. Then, because $\boldsymbol{\nabla}_a V^*(\cdot, y')$ is strictly decreasing on X for any $y' \in Y$,

$$\boldsymbol{\nabla}_a V^*(a^*(a,y_1), y') > \boldsymbol{\nabla}_a V^*(a^*(a,y_2), y')$$

for any $y' \in Y$. As such, we must have

$$\int_Y \boldsymbol{\nabla}_a V^*(a^*(a,y_1), y') d\lambda(y') \geq \int_Y \boldsymbol{\nabla}_a V^*(a^*(a,y_2), y') d\lambda(y'),$$

and the inequality holds as as a strict inequality because otherwise $\boldsymbol{\nabla}_a V^*(a^*(a,y_1), y') =$

$\nabla_a V^*(a^*(a, y_2), y')$ for λ-a.e. $y' \in Y$ by the vanishing property for non-negative functions, a contradiction.

Therefore, by the Euler inequality we have

$$u'[c^*(a, y_1)] \geq \beta(1+r) \int_Y \nabla_a V^*(a^*(a, y_1), y') d\lambda(y')$$

$$> \beta(1+r) \int_Y \nabla_a V^*(a^*(a, y_2), y') d\lambda(y') = u'[c^*(a, y_2)],$$

which implies that

$$c^*(a, y_1) < c^*(a, y_2).$$

In any case, we have $c^*(a, y_1) < c^*(a, y_2)$.

Q.E.D.

7.1.5 Precautionary Saving and Prudence

7.1.6 Boundedness of the Asset Space

We are now interested in the conditions under which asset holdings are bounded above. In other words, we want to find an $\overline{a} \in X$ such that $a \in [\underline{a}, \overline{a}]$ implies $a^*(a,y) \leq \overline{a}$ for any $y \in Y$. We will work toward two results. First, we want to show that, if $\beta(1+r) < 1$, then the asset space is bounded. Next, we want to show that this is a necessary condition for the asset space to be bounded, that is, the asset space is unbounded if $\beta(1+r) \geq 1$. We begin with a preliminary lemma that tells us that consumption increases indefinitely as wealth increases:

Lemma (Unboundedness of Consumption)
Consider the consumption-savings model above. For any $y \in Y$,

$$\lim_{a \to \infty} c^*(a,y) = +\infty.$$

If the endowment shocks are iid, then there must exist some $a \in X$ such that $a^*(a,\overline{y}) > -\phi$, that is, the borrowing constraint cannot always be binding under the best possible endowment.

Proof) Choose any $y \in Y$ and $a > -\phi$. The envelope theorem for the value function then tells us that

$$\nabla_a V^*(a,y) = (1+r)u'\left[c^*(a,y)\right].$$

To show the desired result, we need only show that the right hand side goes to 0 as $a \to \infty$; then, the Inada condition implies that consumption must increase without bound.

To this end, we will make use of the fact that $V^*(\cdot,y)$ is a strictly concave function on $[-\phi,+\infty)$ that is also bounded on that same interval. First, note that $\nabla_a V^*(a,y) > 0$ for any $a > -\phi$ because $1+r > 0$ and $u'(\cdot) > 0$ on $(0,+\infty)$. Now choose any sequence $\{a_n\}_{n \in N_+}$ in $(-\phi,+\infty)$ that increases to $+\infty$, and suppose that there exists some $k > 0$ such that

$$\nabla_a V^*(a_n,y) \geq k$$

for any $n \in N_+$. Fix some $n \in N_+$. Then, by the mean value theorem,

$$V^*(a_n, y) - V^*(a_1, y) = \nabla_a V^*(\tilde{a}, y)(a_n - a_1)$$

for some convex combination $\tilde{a}$ of a_n and a_1. Since $a_n \geq a_1$ and $\nabla_a V^*(\cdot, y)$ is strictly decreasing, it follows that

$$V^*(a_n, y) - V^*(a_1, y) \geq \nabla_a V^*(a_1, y)(a_n - a_1) \geq k(a_n - a_1).$$

This holds for any $n \in N_+$, so sending $n \to \infty$ shows us that

$$\lim_{n \to \infty} V^*(a_n, y) = +\infty,$$

which contradicts the boundedness of V^*. Therefore, it must be the case that, for any $\epsilon > 0$, there always exists some $N \in N_+$ such that

$$\epsilon > \nabla_a V^*(a_N, y) \geq 0.$$

Since $a_n \geq a_N$ for any $n \geq N$ and $\nabla_a V^*(\cdot, y)$ is a strictly decreasing function, this implies that

$$|\nabla_a V^*(a_n, y)| = \nabla_a V^*(a_n, y) < \epsilon$$

for any $n \geq N$. We can find such an $N \in N_+$ for any $\epsilon > 0$, so it follows that

$$\lim_{n \to \infty} \nabla_a V^*(a_n, y) = 0.$$

This, in turn, holds for any sequence $\{a_n\}_{n \in N_+}$ in $(-\phi, +\infty)$ that increases to $+\infty$, so it must be the case that

$$\lim_{a \to \infty} \nabla_a V^*(a, y) = 0.$$

Finally, this implies, via the envelope theorem, that

$$\lim_{a \to \infty} u'[c^*(a, y)] = 0.$$

The Inada condition $\lim_{c \to \infty} u'(c) = 0$ now allows us to conclude that $c^*(a, y) \to$

$+\infty$ as $a \to \infty$.

Now let the endowment shocks be drawn independently from a distribution λ on $(Y, \mathcal{Y})$. Recall that this implies the monotonicity of consumption in endowment.

Suppose, for the sake of contradiction, that $a^*(a, \overline{y}) = -\phi$ for any $a \in X$. Then, by the Euler inequality we have

$$u'[c^*(a, \overline{y})] \geq \beta(1+r) \int_Y u' \left[c^*(-\phi, y') \right] d\lambda(y')$$

for any $a \in X$. Since $c^*(-\phi, y') \leq c^*(-\phi, \overline{y})$ for any $y' \in Y$ by the monotonicity of consumption in endowments, we have

$$u'[c^*(a, \overline{y})] \geq \beta(1+r) u'[c^*(-\phi, \overline{y})] > 0.$$

Taking $a \to \infty$ on both sides now yields

$$0 \geq \beta(1+r) u'[c^*(-\phi, \overline{y})] > 0,$$

a contradiction. Therefore, there must exist some $a \in X$ such that $a^*(a, \overline{y}) > -\phi$.

Q.E.D.

Theorem (Boundedness of Asset Space under $\beta(1+r) < 1$)
Consider the consumption-savings model above. Suppose that $\beta(1+r) < 1$ and the endowment shocks are iid. Assume in addition that $u'(\cdot)$ is twice continuously differentiable and convex. Then, there exists some $\overline{a} > 0$ such that $a^*(a, y) < a$ for any $a \geq \overline{a}$ and $y \in Y$.

Proof) We will show that there exists some $A > 0$ such that $a^*(a, \overline{y}) < a$ for any $a \geq A$. This will immediately imply our desired result, since for any $(a, y) \in X \times Y$ such that $a \geq A$, we will now have

$$a^*(a, y) \leq a^*(a, \overline{y}) < a$$

by the monotonicity of savings in endowment.

Note that $a^*(\cdot, \overline{y})$ is an increasing function on X, so that either $a^*(a, \overline{y})$ converges to some constant $M \in \mathbb{R}$ or it diverges to $+\infty$ as $a \to \infty$. If

$$\lim_{a \to \infty} a^*(a, \overline{y}) = M < +\infty,$$

then for any $a > M$ we can see that

$$a^*(a, \overline{y}) \leq M < a$$

and we are done.

Now suppose that $a^*(\cdot, \overline{y})$ is unbounded, so that

$$\lim_{a \to \infty} a^*(a, \overline{y}) = +\infty.$$

Then, the preceding lemma implies that

$$\lim_{a \to \infty} c^*(a^*(a, \overline{y}), y) = +\infty$$

for any $y \in Y$; this will come in handy later.

Since $a^*(a, \overline{y}) > -\phi$ for any a that is large enough, the Euler equation now holds as

$$\begin{aligned} u'\left[c^*(a, \overline{y})\right] &= \beta(1+r) \int_Y u'\left[c^*(a^*(a, \overline{y}), y')\right] d\lambda(y') \\ &= \beta(1+r) \frac{\int_Y u'\left[c^*(a^*(a, \overline{y}), y')\right] d\lambda(y')}{u'\left[c^*(a^*(a, \overline{y}), \overline{y})\right]} \cdot u'\left[c^*(a^*(a, \overline{y}), \overline{y})\right]. \end{aligned}$$

for large enough a. Our first objective is to show that

$$\lim_{a \to \infty} \frac{\int_Y u'\left[c^*(a^*(a, \overline{y}), y')\right] d\lambda(y')}{u'\left[c^*(a^*(a, \overline{y}), \overline{y})\right]} = 1.$$

To this end, note that, because $c^*(a^*(a, \overline{y}), \cdot)$ is strictly increasing on Y and $u'(\cdot)$ is strictly decreasing on $(0, +\infty)$, we have

$$u'\left[c^*(a^*(a, \overline{y}), \overline{y})\right] \leq u'\left[c^*(a^*(a, \overline{y}), y')\right] \leq u'\left[c^*(a^*(a, \overline{y}), \underline{y})\right]$$

for any $y' \in Y$. It follows that

$$1 \leq \frac{\int_Y u'\left[c^*(a^*(a,\overline{y}),y')\right] d\lambda(y')}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]} \leq \frac{u'\left[c^*(a^*(a,\overline{y}),\underline{y})\right]}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]}.$$

Consider a second order Taylor expansion of $u'(c_2)$ around c_1:

$$u'(c_2) = u'(c_1) + u''(c_1)(c_2 - c_1) + \frac{1}{2}u'''(\tilde{c})(c_2 - c_1)^2$$

for some convex combination $\tilde{c}$ of c_1 and c_2. Since $u'(\cdot)$ is convex, it follows that $u'''(\cdot) \geq 0$ on $(0, +\infty)$, so that

$$u'(c_2) \geq u'(c_1) + u''(c_1)(c_2 - c_1),$$

which implies that

$$\frac{u'(c_2)}{u'(c_1)} \geq 1 + \frac{u''(c_1)c_1}{u'(c_1)}\left(\frac{c_2}{c_1} - 1\right).$$

Since

$$-\frac{u''(c_1)c_1}{u'(c_1)} \leq \mu$$

by assumption, we now have

$$\frac{u'(c_2)}{u'(c_1)} \geq 1 - \mu\left(\frac{c_2}{c_1} - 1\right).$$

Now putting $c_2 = c^*(a^*(a,\overline{y}),\overline{y})$ and $c_1 = c^*(a^*(a,\overline{y}),\underline{y})$, we can see that

$$1 \geq \frac{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]}{u'\left[c^*(a^*(a,\overline{y}),\underline{y})\right]} \geq 1 - \mu\left(\frac{c^*(a^*(a,\overline{y}),\overline{y})}{c^*(a^*(a,\overline{y}),\underline{y})} - 1\right).$$

By the definition of the consumption policy function, we have

$$c^*(a^*(a,\overline{y}),\overline{y}) = \overline{y} + (1+r)a^*(a,\overline{y}) - a^*\left[a^*(a,\overline{y}),\overline{y}\right]$$

$$c^*(a^*(a,\overline{y}),\underline{y}) = \underline{y} + (1+r)a^*(a,\overline{y}) - a^*\left[a^*(a,\overline{y}),\underline{y}\right],$$

which implies that

$$0 \leq c^*(a^*(a,\overline{y}),\overline{y}) - c^*(a^*(a,\overline{y}),\underline{y}) = \overline{y} - \underline{y} - \left(a^*\left[a^*(a,\overline{y}),\overline{y}\right] - a^*\left[a^*(a,\overline{y}),\underline{y}\right]\right).$$

Since $a^*\left[a^*(a,\overline{y}),\cdot\right]$ is increasing on Y, this implies

$$a^*\left[a^*(a,\overline{y}),\overline{y}\right] - a^*\left[a^*(a,\overline{y}),\underline{y}\right] \geq 0$$

and therefore

$$0 \leq c^*(a^*(a,\overline{y}),\overline{y}) - c^*(a^*(a,\overline{y}),\underline{y}) \leq \overline{y} - \underline{y}.$$

By implication,

$$0 \leq \frac{c^*(a^*(a,\overline{y}),\overline{y})}{c^*(a^*(a,\overline{y}),\underline{y})} - 1 \leq \frac{\overline{y} - \underline{y}}{c^*(a^*(a,\overline{y}),\underline{y})}.$$

We showed above that

$$\lim_{a\to\infty} c^*(a^*(a,\overline{y}),\underline{y}) = +\infty;$$

it follows that

$$\lim_{a\to\infty}\left(\frac{c^*(a^*(a,\overline{y}),\overline{y})}{c^*(a^*(a,\overline{y}),\underline{y})} - 1\right) = 0,$$

and therefore that

$$\lim_{a\to\infty} \frac{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]}{u'\left[c^*(a^*(a,\overline{y}),\underline{y})\right]} = 1$$

as well. This shows us that

$$\lim_{a\to\infty} \frac{\int_Y u'\left[c^*(a^*(a,\overline{y}),y')\right] d\lambda(y')}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]} = 1.$$

Define

$$\epsilon = \frac{1}{\beta(1+r)} - 1 > 0,$$

where the positivity of ϵ follows from the assumption that $\beta(1+r) < 1$. By the result we just proved, there exists some $\overline{a} > 0$ such that

$$\left| \frac{\int_Y u'\left[c^*(a^*(a,\overline{y}),y')\right] d\lambda(y')}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]} - 1 \right| < \epsilon,$$

and in particular,

$$\frac{\int_Y u'\left[c^*(a^*(a,\overline{y}),y')\right] d\lambda(y')}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]} < 1+\epsilon,$$

for any $a \geq \overline{a}$. It follows that, for any $a \geq \overline{a}$,

$$\begin{aligned} u'\left[c^*(a,\overline{y})\right] &= \beta(1+r)\frac{\int_Y u'\left[c^*(a^*(a,\overline{y}),y')\right] d\lambda(y')}{u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right]} u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right] \\ &< \beta(1+r)(1+\epsilon)u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right] = u'\left[c^*(a^*(a,\overline{y}),\overline{y})\right], \end{aligned}$$

since $\beta(1+r)(1+\epsilon) = 1$. It follows that

$$c^*(a,\overline{y}) > c^*(a^*(a,\overline{y}),\overline{y}),$$

and as such,

$$a^*(a,\overline{y}) < a,$$

for any $a \geq \overline{a}$ by the strict monotonicity of the consumption policy function in wealth.

Q.E.D.

Theorem (Unboundedness of Asset Space under $\beta(1+r) \geq 1$)
Consider the consumption-savings model above. Suppose that $\beta(1+r) \geq 1$ and that the endowment shocks are iid. Then, for any $\overline{a} \in X$ there exists some $a \geq \overline{a}$ and $y \in Y$ such that $a^*(a,y) \geq a$.

Proof) Suppose, for the sake of contradiction, that there exists some $\overline{a} \in X$ such that $a^*(a,y) < a$ for any $a \geq \overline{a}$ and $y \in Y$. Define the continuous function $\overline{h} : X \to \mathbb{R}$ as

$$\overline{h}(a) = a - a^*(a, \overline{y}) \quad \text{for any } a \in X.$$

Since

$$\overline{h}(-\phi) = -\phi - a^*(-\phi, \overline{y}) \leq 0$$

and

$$\overline{h}(\overline{a}) = \overline{a} - a^*(\overline{a}, \overline{y}) > 0,$$

it follows that

$$\overline{h}(-\phi) \leq 0 < \overline{h}(\overline{a}).$$

If $\overline{h}(-\phi) = 0$, then $-\phi$ is a fixed point of $a^*(\cdot, \overline{y})$. On the other hand, if $\overline{h}(-\phi) < 0$, then by the intermediate value theorem, there exists some $A \in X$ such that $\overline{h}(A) = 0$, or equivalently,

$$a^*(A, \overline{y}) = A.$$

In any case, $a^*(\cdot, \overline{y})$ has a fixed point in $X = [-\phi, +\infty)$.

Now, the Euler inequality implies that

$$u'\left[c^*(A, \overline{y})\right] \geq \beta(1+r) \int_Y u'\left[c^*(A, y')\right] d\lambda(y').$$

Since $\beta(1+r) \geq 1$, we have

$$u'\left[c^*(A, \overline{y})\right] \geq \int_Y u'\left[c^*(A, y')\right] d\lambda(y'),$$

and because $c^*(A,\cdot)$ is strictly monotonic on Y, it follows that $c^*(A,y') \leq c^*(A,\overline{y})$ and thus

$$u'\left[c^*(A,y')\right] \geq u'\left[c^*(A,\overline{y})\right]$$

for any $y' \in Y$. By implication,

$$\int_Y u'\left[c^*(A,y')\right] d\lambda(y') \geq \int_Y u'\left[c^*(A,\overline{y})\right] d\lambda(y') = u'\left[c^*(A,\overline{y})\right].$$

Suppose that this inequality holds as an equality. Then, by the vanishing property for non-negative functions, we would have $u'[c^*(A,y')] = u'[c^*(A,\overline{y})]$, and therefore

$$c^*(A,y') = c^*(A,\overline{y})$$

for λ-a.e. $y' \in Y$. This contradicts the fact that λ places positive probability on $\{\underline{y}\}$ and $c^*(A,\underline{y}) < c^*(A,\overline{y})$. Therefore, the inequality above must be strict.

We have thus shown that

$$u'\left[c^*(A,\overline{y})\right] \geq \int_Y u'\left[c^*(A,y')\right] d\lambda(y') > u'\left[c^*(A,\overline{y})\right],$$

which is a contradiction. Therefore, there cannot exist any $\overline{a} \in X$ such that $a^*(a,y) < a$ for every $a > \overline{a}$ and $y \in Y$.

Q.E.D.

7.1.7 The Stationary Distribution of Savings

Using the machinery we have developed so far, we can establish that there exists a unique stationary distribution of savings that the economy converges to regardless of how the initial distribution is specified.

First, we define the Markov kernel associated with the policy process $\{a_t, y_t\}_{t\in\mathbb{N}}$. Specifically, let Q be the Markov kernel on $(S,\mathcal{S}) = (X,\mathcal{X})\bigotimes(Y,\mathcal{Y})$ defined as

$$Q((a,y),B) = \int_Y I_B(a^*(a,y),y')P(y,dy')$$

for any $(a,y) \in S$ and $B \in \mathcal{S}$. Because P possess the Feller property, we can see that Q does as well. Furthermore, if the endowment shocks are independently generated, so that $P(y,\cdot) = \lambda$ for any $y \in Y$ and some distribution λ on $(Y,\mathcal{Y})$, then $a^*(\cdot)$ is increasing on S and Q is a monotonic kernel (note that λ is technically a monotonic Markov kernel as well).

To establish the existence of a unique stationary distribution for Q to which it converges weakly, it remains to establish that Q satisfies the stable mixing property. Specifically, we want to show that

- There exists some closed 2-cell $[c,d]$ contained in S such that $[c,d]$ is a stable set for Q; $Q(s,[c,d]) = 1$ for any $s \in [c,d]$.
- Q eventually enters $[c,d]$; for any distribution μ_0 on $(S,\mathcal{S})$,

$$\lim_{t\to\infty}(\mu_0 Q^t)([c,d]) = 1.$$

- Q satisfies the splitting condition on $[c,d]$; there exists a $\hat{s} \in [c,d]$, $\epsilon > 0$ and $N \in N_+$ such that

$$Q^N(c,[c,\hat{s}]) \geq \epsilon, \quad Q^N(d,[\hat{s},d]) \geq \epsilon.$$

We verify that the first two conditions hold in the next theorem, and prove that the splitting condition holds after that.

Theorem (Existence of a Stable Set for Policy Process)
Consider the consumption-savings model above. Suppose that $\beta(1+r) < 1$ and that the endowment shocks are drawn iid from the distribution λ. Then, there exists a stable set $[c,d]$ for the policy kernel Q defined above. Furthermore, for any distribution μ_0 on $(S,\mathcal{S})$,

$$\lim_{t\to\infty} (\mu_0 Q^t)([c,d]) = 1.$$

Proof) Let λ be the distribution on $(Y,\mathcal{Y})$ from which the endowment shocks are drawn.From the boundedness result we obtained above, there exists some $A > 0$ such that

$$a^*(a,y) < a \quad \text{for any } a \geq A \text{ and } y \in Y.$$

By implication, if $a \leq A$, then

$$a^*(a,y) \leq a^*(A,y) < A$$

for any $y \in Y$ by the monotonicity of the savings policy function. Therefore, we can define

$$c = (-\phi, \underline{y}), \quad d = (A, \overline{y});$$

then, $[c,d]$ is a stable set for the kernel Q, since for any $s = (a,y) \in [c,d] = [-\phi, A] \times Y$,

$$Q(s,[c,d]) = I_{[-\phi,A]}(a^*(a,y)) \cdot \lambda(Y) = 1.$$

We now proceed in steps to show that the stable set $[c,d]$ is reached with probability approaching 1 for any Markov chain with kernel Q. First, we show that any chain started from some fixed initial point on $X \times Y$ with kernel Q enters $[c,d]$ with probability approaching 1. Afterward, we use this result to prove that the same result holds for any Markov chain with kernel Q and arbitrary initial distribution.

Reaching the Stable Set from a Deterministic Initial Point

Choose any $s_0 = (a_0, y_0) \in X \times Y$. If $a_0 \leq A$, then

$$Q^t(s_0, [c,d]) = 1$$

for any $t \geq 1$ by the fact that $[c,d]$ is a stable set for Q.

Now suppose that $a_0 > A$. We want to find out if

$$\lim_{t\to\infty} Q^t(s_0, [c,d]) = 1$$

in this case as well. To this end, let $\{(A_t, Y_t)\}_{t\in\mathbb{N}}$ be a sequence of random variables that are generated as follows:

$$(A_0, Y_0) = (a_0, y_0), \quad Y_t \sim \lambda, \quad A_t = a^*(A_{t-1}, Y_{t-1}) \quad \text{for any } t \geq 1.$$

Then, $\{(A_t, Y_t)\}_{t\in\mathbb{N}}$ is a Markov chain with kernel Q and initial distribution $\delta_{(a_0,y_0)}$. Since the distribution of (A_t, Y_t) is given as $Q^t(s_0, \cdot)$, we want to show that

$$\lim_{T\to\infty} \mathbb{P}((A_T, Y_T) \in [c,d]) = 1.$$

To this end, choose a single outcome $\omega \in \Omega$ in the sample space; we focus on the (deterministic) path $\{A_t(\omega), Y_t(\omega)\}_{t\in\mathbb{N}}$, which we denote by $\{a_t, y_t\}_{t\in\mathbb{N}}$.

Suppose that $a_t > A$ for every $t \geq 1$. Then, because $a^*(a,y) < a$ for any $y \in Y$ and $a > A$, it follows that $\{a_t, y_t\}_{t\in\mathbb{N}}$ is contained in the compact set $[A,\ a_0] \times Y$, and that

$$a_{t+1} = a^*(a_t, y_t) < a_t$$

for any $t \in \mathbb{N}$. Since a decreasing function always converges, we know that $\{a_t\}_{t\in\mathbb{N}}$

converges to some $a^L \in X$ such that $A \leq a^L$. Since

$$a_{t+1} = a^*(a_t, y_t) \leq a^*(a_t, \overline{y}) < a_t,$$

taking $t \to \infty$ on both sides yields

$$a^L \leq a^*(a^L, \overline{y}) \leq a^L$$

so that $a^*(a^L, \overline{y}) = a^L$ and $a^L \geq A$. But by the way that A was chosen, we must have $a^*(a^L, \overline{y}) < a^L$, which contradicts our finding. Therefore, there must be a $T \geq 1$ such that $a_T \leq A$.

In this case, because

$$a^*(a, y) \leq a^*(A, y) < A$$

for any $(a, y) \in [c, d]$, it follows that $(a_t, y_t) \in [c, d]$ for any $t \geq T$. By implication,

$$I_{[c,d]} \circ (A_t(\omega), Y_t(\omega)) \to 1$$

as $t \to \infty$.

We have just shown that

$$I_{[c,d]} \circ (A_t, Y_t) \overset{a.s.}{\to} 1;$$

indeed, we proved that this convergence occurs everywhere on the sample space Ω. By the DCT, we can now conclude that

$$\lim_{t \to \infty} \mathbb{P}((A_t, Y_t) \in [c, d]) = 1,$$

which is equivalent to saying that

$$\lim_{t \to \infty} Q^t(s_0, [c, d]) = 1.$$

Reaching the Stable Set from an Initial Distribution

Now let μ_0 be any arbitrary distribution on $(Y, \mathcal{Y})$. We want to show that

$$\lim_{t\to\infty}(\mu_0 Q^t)([c,d]) = 1.$$

To this end, define the disjoint collection of measurable sets $\{B_n\}_{n\in\mathbb{N}}$ as

$$B_0 = [-\phi, 0)\times Y, \quad B_n = [n-1, n)\times Y$$

for any $n \in \mathbb{N}$. Then, we can see that $\bigcup_n B_n = X \times Y$, so that the countable additivity of indefinite integrals implies that

$$\begin{aligned}(\mu_0 Q^t)([c,d]) &= \int_{X\times Y} Q^t(s,[c,d])d\mu_0(s)\\ &= \sum_{n=0}^{\infty}\int_{B_n} Q^t(s,[c,d])d\mu_0(s).\end{aligned}$$

The monotonicity of the Markov kernel Q^t indicates that, because $s \le (n,\overline{y})$ for any $s \in B_n$ and the indicator function $I_{(A,+\infty)\times Y}$ is increasing on S,

$$Q^t(s,(A,+\infty)\times Y) \le Q^t((n,\overline{y}),\ (A,+\infty)\times Y)$$

for any $s \in B_n$. It follows that

$$Q^t((n,\overline{y}),\ [c,d]) = 1 - Q^t((n,\overline{y}),\ (A,+\infty)\times Y) \le 1 - Q^t(s,(A,+\infty)\times Y) = Q^t(s,\ [c,d])$$

for any $s \in B_n$ and thus

$$\begin{aligned}1-(\mu_0 Q^t)([c,d]) &= 1 - \sum_{n=0}^{\infty}\int_{B_n} Q^t(s,[c,d])d\mu_0(s)\\ &\le 1 - \sum_{n=0}^{\infty} Q^t((n,\overline{y}),\ [c,d])\mu_0(B_n).\end{aligned}$$

Note that

$$1 = \mu_0(X\times Y) = \sum_{n=0}^{\infty}\mu_0(B_n)$$

by the countable additivity of probability measures. For any $\epsilon > 0$, there exists some $N \in N_+$ such that

$$1 - \sum_{n=0}^{N} \mu_0(B_n) < \frac{\epsilon}{2},$$

so that

$$\begin{aligned}
1 - (\mu_0 Q^t)([c,d]) &\leq 1 - \sum_{n=0}^{\infty} Q^t((n,\overline{y}),\, [c,d])\mu_0(B_n) \\
&\leq 1 - \sum_{n=0}^{N} Q^t((n,\overline{y}),\, [c,d])\mu_0(B_n) \\
&= \left(1 - \sum_{n=0}^{N} \mu_0(B_n)\right) + \sum_{n=0}^{N} \left(1 - Q^t((n,\overline{y}),\, [c,d])\right) \mu_0(B_n) \\
&< \frac{\epsilon}{2} + \sum_{n=0}^{N} \left(1 - Q^t((n,\overline{y}),\, [c,d])\right).
\end{aligned}$$

Applying monotonicity again shows us that

$$Q^t((n,\overline{y}),\, [c,d]) \geq Q^t((N,\overline{y}),\, [c,d])$$

for any $0 \leq n \leq N$, so

$$1 - (\mu_0 Q^t)([c,d]) < \frac{\epsilon}{2} + N\left(1 - Q^t((N,\overline{y}),\, [c,d])\right)$$

for any $t \geq 1$.

From what we proved earlier,

$$Q^t((N,\overline{y}),\, [c,d]) \to 1$$

as $t \to \infty$. Thus, there exists a $T \in N_+$ such that

$$1 - Q^t((N,\overline{y}),\, [c,d]) < \frac{\epsilon}{2N}$$

for any $t \geq T$, so that

$$1 - (\mu_0 Q^t)([c,d]) < \epsilon$$

for any $t \geq T$. In this case, T depends only on N and ϵ, and N in turn depends only on ϵ, so it follows that T depends only on ϵ.

We can find such a $T \in N_+$ for any $\epsilon > 0$, so by definition

$$\lim_{t \to \infty} (\mu_0 Q^t)([c,d]) = 1$$

and we are done.

Q.E.D.

Theorem (Splitting Condition for the Policy Process)
Consider the consumption-savings model above. Suppose that $\beta(1+r) < 1$ and that the endowment shocks are drawn iid from the distribution λ. Let $[c,d]$ be the stable set found in the preceding theorem, and assume $a^*(-\phi, \overline{y}) > -\phi$ and that $\lambda(\{\overline{y}\}), \lambda(\{\underline{y}\}) > 0$. Then, there exists some $\hat{s} \in [c,d]$, $N \in N_+$ and $\epsilon > 0$ such that

$$Q^N(c, [\hat{s}, d]) \geq \epsilon, \quad Q^N(d, [c, \hat{s}]) \geq \epsilon.$$

Proof) Let $[c,d] = [-\phi, A] \times Y$ be the stable set defined in the previous theorem, where $A > 0$ satisfies the condition that $a^*(a,y) < a$ for any $(a,y) \in [A, +\infty) \times Y$. Define the continuous functions $\overline{h}, \underline{h} : X \to \mathbb{R}$ as

$$\overline{h}(a) = a - a^*(a, \overline{y}), \quad \underline{h} = a - a^*(a, \underline{y})$$

for any $a \in X$. We first focus on $\overline{h}$. Since

$$\overline{h}(-\phi) = -\phi - a^*(-\phi, \overline{y}) < 0$$

by assumption and

$$\overline{h}(A) = A - a^*(A, \overline{y}) > 0,$$

by our choice of A, it follows that

$$\overline{h}(-\phi) < 0 < \overline{h}(\overline{a}).$$

By the intermediate value theorem, there exists some $a \in X$ such that $\overline{h}(a) = 0$, or equivalently,

$$a^*(a, \overline{y}) = a.$$

Therefore, the set $\overline{H}$ of fixed points defined as

$$\overline{H} = \{a \in X \mid \overline{h}(a) = a\}$$

is non-empty and bounded below by $-\phi$ and above by A. Furthermore, $\overline{H}$ is a closed set; if $\{a_n\}_{n \in N_+}$ is a sequence of fixed points of $\overline{h}$ that converges to some $a \in X$, then the continuity of $\overline{h}(\cdot)$ and the fact that $\overline{h}(a_n) = a_n$ for any $n \in N_+$ implies that $\overline{h}(a) = a$, so that $a \in \overline{H}$. It follows that

$$\overline{a} = \inf \overline{H} \in \overline{H}, \quad \overline{a}_M = \sup \overline{H} \in \overline{H},$$

and that $-\phi < \overline{a}_m \leq \overline{a}_M < A$.

Turning our attention to $\underline{h}(\cdot)$, let $\underline{H}$ be the set of its fixed points:

$$\underline{H} = \{a \in X \mid \underline{h}(a) = a\}.$$

By the same argument as the one for $\overline{h}(\cdot)$, we can see that $\underline{H}$ is non-empty. Choose any $a^f \in \underline{H}$, so that $\underline{h}(a^f) = a^f$, and suppose $a^f > -\phi$. Then, the Euler inequality holds as an equality, and is given as

$$u'\left[c^*(a^f, \underline{y})\right] = \beta(1+r) \int_Y u'\left[c^*(a^f, y')\right] d\lambda(y').$$

By the strict monotonicity of consumption in endowments, $u'\left[c^*(a^f, y')\right] \leq$

$u'\left[c^*(a^f, \underline{y})\right]$ for any $y' \in Y$, so that

$$u'\left[c^*(a^f, \underline{y})\right] \leq \beta(1+r)u'\left[c^*(a^f, \underline{y})\right] < u'\left[c^*(a^f, \underline{y})\right],$$

where the final inequality follows from the assumption that $\beta(1+r) < 1$. This is clearly a contradiction, so it must be the case that $\underline{H} = \{-\phi\}$.

We have found that the (unique) fixed point of $\underline{h}(\cdot)$ is smaller than the smallest fixed point of $\overline{h}(\cdot)$. We now define

$$\hat{s} = \left(\frac{-\phi + \overline{a}_m}{2}, \underline{y}\right) \in [c, d].$$

Consider first a household that starts the economy with the initial wealth and endowment $d = (A, \overline{y})$ and receives endowment shocks equal to $\underline{y}$ from each period then on. Because $-\phi$ is the unique fixed point for $\underline{h}(\cdot)$, we can see that

$$\underline{h}(a) = a^*(a, \underline{y}) < a$$

for any $-\phi < a < A$[1]. Consider the sequence $\{a_t\}_{t\in\mathbb{N}}$ constructed as $a_0 = A$, $a_1 = a^*(A, \overline{y}) < A$ and

$$a_{t+1} = a^*(a_t, \underline{y}) \quad \text{for any } t \geq 1.$$

Since $\{a_t\}_{t\in\mathbb{N}} \subset [-\phi, A)$, it follows that

$$a_{t+1} = a^*(a_t, \underline{y}) \leq a_t$$

for any $t \in N_+$. $\{a_t\}_{t\in N_+}$ is a monotonically decreasing sequence and thus convergent to a limit that is at least as large as $-\phi$; denote this limit by $a_L \geq -\phi$. Since

$$a_{t+1} = a^*(a_t, \underline{y}) \leq a_t$$

[1]Suppose $a^*(a, \underline{y}) > a$ for some $-\phi < a < A$.Then, because $a^*(A, \underline{y}) < A$, it follows that there must exist some fixed point of $\underline{h}(\cdot)$ between a and A, which contradicts the fact that $-\phi$ is the unique fixed point of $\underline{h}(\cdot)$

for any $t \in N_+$, taking $t \to \infty$ on both sides implies that

$$a_L = -a^*(a_L, \underline{y}) \leq a_L,$$

or in other words that $a^*(a_L, \underline{y}) = a_L$. Thus, a_L is a fixed point of $\underline{h}(\cdot)$, which immediately tells us that $a_L = -\phi$. In light of the fact that $-\phi < \hat{s}$, there exists an $N_1 \in N_+$ such that

$$-\phi < a_t < \hat{s}$$

for any $t \geq N_1$. In other words,

$$Q^t(d, [c, \hat{s}]) \geq \lambda(\{\underline{y}\})^t$$

for any $t \geq N_1$, that is, if we receive an endowment shock equal to $\underline{y}$ for at least N_1 periods, then the Markov chain started at d will enter $[c, \hat{s}]$.

Now consider a household that starts with initial wealth and endowment $c = (-\phi, \underline{y})$ and continually receives endowment shocks equal to $\overline{y}$. We generate the sequence $\{a_t\}_{t \in \mathbb{N}}$ as $a_0 = -\phi$, $a_1 = a^*(-\phi, \underline{y}) = -\phi$ and

$$a_{t+1} = a^*(a_t, \overline{y}) \quad \text{for any } t \geq 1.$$

We know that $a_1 = -\phi < \overline{a}_m$. Now suppose that $a_t \leq \overline{a}_m$ for some $t \geq 1$. Then, by the monotonicity of the savings function,

$$a_{t+1} = a^*(a_t, \overline{y}) \leq a^*(\overline{a}_m, \overline{y}) = \overline{a}_m$$

as well. By induction, $a_t \leq \overline{a}_m$ for any $t \in \mathbb{N}$.

Since $a^*(a, \overline{y}) > a$ for any $a < \overline{a}_m$[2], it follows that

$$a_t < a^*(a_t, \overline{y}) = a_{t+1}$$

for any $t \in N_+$. It follows that $\{a_t\}_{t \in N_+}$ is a strictly increasing sequence with

[2]Suppose that $a^*(a, \overline{y}) < a$ for some $a < \overline{a}_m$. Then, because $a^*(\underline{a}, \overline{y}) > \underline{a}$, this means that there exists some fixed point of $\overline{h}(\cdot)$ smaller than $\overline{a}_m$, which contradicts the fact that $\overline{a}_m$ is the infimum of $\overline{H}$.

upper bound $\overline{a}_m$ and therefore converges to some limit $a^L \leq \overline{a}_m$. Taking $t \to \infty$ in the inequality

$$a_t < a^*(a_t, \overline{y}) = a_{t+1}$$

shows us that $a^L = a^*(a^L, \overline{y})$, so that $a^L \in \overline{H}$. Since a^L is smaller than or equal to $\overline{a}_m$, and $\overline{a}_m$ is the smallest fixed point of $\overline{h}(\cdot)$, it follows that $\overline{a}_m = a^L$. Thus, there exists some $N_2 \in N_+$ such that

$$\hat{s} < a_t < \overline{a}_m$$

for any $t \geq N_2$, which implies that

$$Q^t(c,\, [\hat{s}, d]) \geq \lambda(\{\overline{y}\})^t$$

for any $t \geq N_2$.

Putting these results together, letting $N = \mathsf{max}(N_1, N_2) \in N_+$ and

$$\epsilon = \mathsf{min}(\lambda(\{\overline{y}\})^N,\, \lambda(\{\underline{y}\})^N) > 0,$$

we can see that

$$Q^N(d,\, [c, \hat{s}]) \geq \epsilon, \quad Q^N(c,\, [\hat{s}, d]) \geq \epsilon.$$

This completes the proof.

Q.E.D.

We have thus shown that Q is a Markov kernel on $(S, \mathcal{S})$ that:

- Satisfies the Feller property
- Satisfies the monotonicity property
- Admits a stable set $[c, d] \subset S$

- Satisfies the splitting condition on $[c, d]$
- Enters $[c, d]$ with probabiliy approaching 1 given any initial distribution.

Therefore, we can conclude that there exists a unique stationary distribution μ^* of Q that assigns all its mass to the stable set $[c, d] = [-\phi, A] \times Y$ and any Markov chain with kernel Q converges weakly to μ^*. One final step that we can take is to investigate exactly where the stationary distribution μ^* assigns all its mass, since the endpoint A of the stable set was chosen as any positive real such that $a^*(a, y) < a$ for any $(a, y) \in [A, +\infty) \times Y$, of which there are infinitely many.

Theorem (Support of Stationary Distribution)
Consider the consumption-savings model above. Suppose that $\beta(1+r) < 1$ and that the endowment shocks are drawn iid from the distribution λ. Assume in addition $a^*(-\phi, \overline{y}) > -\phi$ and that $\lambda(\{\overline{y}\}), \lambda(\{\underline{y}\}) > 0$.. Then, the Markov kernel Q admits a unique stationary distribution μ^* to which any Markov chain with kernel Q converges weakly, and

$$\mu^*([-\phi, \overline{a}_m] \times Y) = 1,$$

where $\overline{a}_m > -\phi$ is the smallest fixed point of the function $\overline{h}(\cdot) = a^*(\cdot, \overline{y})$.

Proof) We use the fact that any Markov chain with kernel Q converges weakly to the stationary distribution μ^*. Choose any $s_0 = (a_0, y_0) \in [-\phi, \overline{a}_m] \times Y$. Note that, for any $(a, y) \in [-\phi, \overline{a}_m] \times Y$, we have

$$a^*(a, y) \leq a^*(\overline{a}_m, \overline{y}) = \overline{a}_m,$$

so that $[-\phi, \overline{a}_m] \times Y$ is another stable set of the kernel Q. Therefore,

$$Q^t(s_0, [-\phi, \overline{a}_m] \times Y) = 1$$

for any $t \in N_+$. Letting $\mu_0 = \delta_{s_0}$, that is, the Dirac measure sitting at s_0, we can see that

$$(\mu_0 Q^t)([-\phi, \overline{a}_m] \times Y) = Q^t(s_0, [-\phi, \overline{a}_m] \times Y) = 1$$

for any $t \in N_+$, and that

$$\mu_0 Q^t \to \mu^* \quad \text{weakly as } t \to \infty.$$

Since $[-\phi, \overline{a}_m] \times Y$ is a closed subset of $X \times Y$, by the Portmanteau theorem we have

$$1 = \limsup_{t \to \infty} (\mu_0 Q^t)([-\phi, \overline{a}_m] \times Y) \leq \mu^*([-\phi, \overline{a}_m] \times Y),$$

which completes our proof.

Q.E.D.

7.2 The Aiyagari Model

The income fluctuation problem in the preceding section serves as a building block for the Aiyagari model, which is populated with a continuum of households that each face an idiosyncratic productivity shock. This induces household-level heterogeneity that produces implications that cannot be replicated in a representative agent framework.

We first consider an economy in which there are no aggregate risks. The only soruce of uncertainty in this model comes from individual productivity shocks that follow the same Markov process iid across individuals, so that, by the law of large numbers, this Markov process can also be used to describe the dynamics of productivity shocks in the macroeconomy across time. In this case, we can think of an equilibrium in which macroeconomic aggregates are constant and the same distribution of wealth and productivity across individuals replicates itself across time. Such equilibria are called stationary equilibria, and this type of model with individual but no aggregate uncertainty is called the canonical Aiyagari model.

7.2.1 Model Assumptions

We make the following formal assumptions:

Households There is a continuum of identical households of measure 1, indexed by $[0,1]$. For simplicity, we assume that the households have fixed, inelastic labor supply equal to 1. Household i is subject to an idiosyncratic productivity shock ϵ_{it} that takes values in the compact measurable subspace $(E,\mathcal{E})$ of $\mathbb{R}$ and whose evolution is determined by the Markov kernel P on $(E,\mathcal{E})$. By the compactness of E, there exist $\underline{\epsilon},\overline{\epsilon} \in E$ such that

$$\underline{\epsilon} = \min E, \quad \overline{\epsilon} = \max E.$$

A household with productivity shock ϵ provides ϵ units of effective labor. We assume that P has a stationary distribution π^*, and that the distribution of effective labor across households is always π^*: therefore, aggregate effective labor supply at any time is fixed at

$$\bar{H} = \int_Z \epsilon d\pi^*(\epsilon).$$

At each period, the household with productivity shock ϵ provides ϵ units of effective labor in return for wages w_t. It can also invest in a single asset (a bond) with rate of return is r_t. This asset is explicitly non-state contingent, so that markets are incomplete. Furthermore, the household faces a borrowing constraint $a_t \geq -\phi$. We let the state space be $X = [-\phi, +\infty)$, and let $\mathcal{X}$ be the corresponding Borel σ-algebra. Denote $(S, \mathcal{S}) = (X, \mathcal{X}) \bigotimes (Z, \mathcal{Z})$.

Preferences The representative consumer's flow utility function is $u : \mathbb{R}_+ \to \mathbb{R}$. u is assumed to be twice continuously differentiable on $(0, +\infty)$ with derivatives

$$u'(\cdot) > 0, \quad u''(\cdot) < 0$$

on $\mathbb{R}$, so that u is strictly concave on $\mathbb{R}$. We assume in addition that the Inada conditions hold:

$$\lim_{c \downarrow 0} u'(c) = +\infty, \quad \lim_{c \to \infty} u'(c) = 0.$$

Finally, we assume that there exists an upper bound on the coefficient for relative risk aversion: there exists a $\mu > 0$ such that

$$0 < -c \frac{u''(c)}{u'(c)} \leq \mu$$

for any $c > 0$.

For computational purposes, we also make the (strong) assumption that u is a bounded function. This assumption is actually violated in many cases by some frequently used utility functions (such as log or CRRA utility) but is satisfied by utility functions that have an exponential form.

The household discounts future utility by the discount factor $\beta \in (0, 1)$.

Technologies There is a representative firm that uses effective labor and capital as its output. Capital is procured in the asset market, where the firm issues bonds

that the households purchase; we are explicit in the fact that these bonds are not zero net supply. The firm also purchases the effective labor of the households.

The firm's production function is given as $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which we assume is CRS with the usual properties

$$F_k(\cdot) > 0, \quad F_h(\cdot) > 0, \quad F_{kk}(\cdot) < 0,$$

where the subscript k represents partial derivation with respect to capital and h with respect to effective labor (human capital).We also assume that $F_h(\cdot, 1)$ is strictly increasing on $(0, +\infty)$, that is, the marginal product of effective labor increases in capital, and that

$$\lim_{k \downarrow 0} F_h(k, 1) = 0, \quad \lim_{k \to \infty} F_h(k, 1) = +\infty.$$

Defining $f(k) = F(k, 1)$ and $f_k(k) = F_k(k, 1)$ for any $k \in (0, +\infty)$, $f(\cdot)$ and $f_k(\cdot)$ are continuous functions, and $f_k(\cdot)$ is strictly decreasing on $\mathbb{R}_+$. We further assume that the Inada conditions hold:

$$\lim_{k \downarrow 0} f_k(k) = +\infty, \quad \lim_{k \to \infty} f_k(k) = 0.$$

This demonstrates that $f_k : (0, +\infty) \to (0, +\infty)$ is a continuous and strictly decreasing bijection from $(0, +\infty)$ into $(0, +\infty)$, so that it admits an inverse function $f_k^{-1}(\cdot)$ on $(0, +\infty)$ that is also continuous and strictly decreasing.

Finally, we assume that capital depreciates at the rate $\delta \in (0, 1)$ each period, and that aggregate consumption and investment add up to output each period.

Factor Prices Wages and interest rates at time t are functions of the aggregate state variable, or aggregate capital, at time t, K_t. Because there is no aggregate uncertainty, they are functions only of the aggregate state variable.

Given the above assumptions, household i's time t flow budget constraint can be written as

$$c_{it} + a_{i,t+1} = w_t \epsilon_{it} + (1 + r_t) a_{it},$$

provided its supplies ϵ_{it} units of effective labor, and is subject to the borrowing constraint $a_{i,t+1} \geq -\phi$.As such, analogously to the stochastic neoclassical growth model, the functional equation corresponding to the household's problem is

$$V(a, \epsilon; K) = \sup_{a' \in [-\phi, \mathcal{W}(K)\epsilon + (1+\mathcal{R}(K))a]} \left[u\left[\mathcal{W}(K)\epsilon + (1 + \mathcal{R}(K))a - a'\right] + \beta \cdot \int_E V(a', \epsilon'; \hat{\varphi}(K)) P(\epsilon, d\epsilon') \right],$$

or more simply,

$$\begin{aligned} V(a, \epsilon; K) = & \max_{(c, a') \in [0, +\infty) \times \mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a', \epsilon'; \hat{\varphi}(K)) P(\epsilon, d\epsilon') \right] \\ \text{subject to} \quad & c + a' = \mathcal{W}(K)\epsilon + (1 + \mathcal{R}(K))a, \\ & a' \geq -\phi, \end{aligned}$$

where $\hat{\varphi}(\cdot)$ is the perceived law of motion for the aggregate state variable K.

7.2.2 Recursive Competitive Equilibrium

Our first instinct is to solve for RCE as we did in the preceding model. That is, we find optimal policy functions $c^*(\cdot)$ and $a^*(\cdot)$ and recover $\hat{\varphi}(\cdot)$ that way. However, we quickly run into a serious problem. In order for agents to consistently forecast prices, they must be able to forecast the aggregate state variable K. In a representative agent setting, this is trivial because the representative household's wealth is equal to K. However, because different households are subject to potentially different productivity shocks in the Aiyagari model, today's aggregate capital K is actually given as

$$K = \int_{X \times E} a d\lambda(a, \epsilon),$$

where λ is a distribution on $X \times E$ that represents the cross-sectional distribution of wealth and productivity across households. Now, next period's capital $K' = \hat{\varphi}(K)$ will

be determined by tomorrow's distribution of wealth and productivity λ':

$$K' = \int_{X \times E} a d\lambda'(a, \epsilon) = \hat{\varphi}(K).$$

In other words, in order for households to hold rational belief, that is, in order for them to be able to perfectly predict today and tomorrow's macroeconomic aggregates, they must know the entire cross-sectional distribution λ of wealth and productivity for both today and tomorrow. This means that the distribution λ, not aggregate capital K, serves as our aggregate state for this model, since knowing the distribution immediately means that we know aggregate capital. Furthermore, we must also be aware of the law of motion for this distribution.

With this finding in mind, we can correctly reformulate the household's problem as

$$\begin{aligned} V(a, \epsilon; \lambda) = & \max_{(c, a') \in [0, +\infty) \times \mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a', \epsilon'; \Psi(\lambda)) P(\epsilon, d\epsilon') \right] \\ \text{subject to} \quad & c + a' = \mathcal{W}(\lambda)\epsilon + (1 + \mathcal{R}(\lambda))a, \\ & a' \geq -\phi, \end{aligned}$$

where $\Psi(\cdot)$ is the law of motion for the cross-sectional distribution of wealth and productivity. Note how the wage and interest rate now depend on the distribution, which determines the aggregate supply of capital.

In this setting, we can define recursive competitive equilibrium as the collection of:

- A value function $V^* : S \times \mathcal{M}_p(\mathcal{S}) \to \mathbb{R}$
- Policy functions $c^* : S \times \mathcal{M}_p(\mathcal{S}) \to \mathbb{R}_+$ and $a^* : S \times \mathcal{M}_p(\mathcal{S}) \to X$ for consumption and savings
- Price functions $\mathcal{W}, \mathcal{R} : \mathcal{M}_p(\mathcal{S}) \to \mathbb{R}_+$
- The firm's capital demand function $K^f : \mathcal{M}_p(\mathcal{S}) \to X$
- A perceived law of motion $\Psi : \mathcal{M}_p(\mathcal{S}) \to \mathcal{M}_p(\mathcal{S})$ for the aggregate state

that satisfy the following conditions:

i) **Household Optimality**
Each household optimizes given the perceived law of motion for the aggregate

state. Specificcally, V^* solves the functional equation

$$V(a,\epsilon;\lambda) = \max_{(c,a')\in[0,+\infty)\times\mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a',\epsilon';\Psi(\lambda))P(\epsilon,d\epsilon')\right]$$

$$\text{subject to} \quad c + a' = \mathcal{W}(\lambda)\epsilon + (1+\mathcal{R}(\lambda))a,$$

$$a' \geq -\phi.$$

$a^* : S \times \mathcal{M}_p(\mathcal{S}) \to X$ is the associated savings policy function, and $c^* : S \times \mathcal{M}_p(\mathcal{S}) \to \mathbb{R}_+$ is defined as

$$c^*(a,\epsilon;\lambda) = \mathcal{W}(\lambda)\epsilon + (1+\mathcal{R}(\lambda))a - a^*(a,\epsilon;\lambda)$$

for any $(a,\epsilon,\lambda) \in S \times \mathcal{M}_p(\mathcal{S})$.

ii) **Profit Maximization**

For any $\lambda \in \mathcal{M}_p(\mathcal{S})$, the firm chooses $K^f(\lambda)$ in a manner that satisfies the first order conditions of profit maximizaiton:

$$\mathcal{W}(\lambda) = F_h(K^f(\lambda),\bar{H}), \quad \mathcal{R}(\lambda) + \delta = F_k(K^f(\lambda),\bar{H}).$$

iii) **Market Clearing**

For any $\lambda \in \mathcal{M}_p(\mathcal{S})$, all markets clear:

$$\bar{H} = \int_{X\times E} \epsilon d\lambda(a,\epsilon)$$

(Labor Market Clearing)

$$K^f(\lambda) = \int_{X\times E} a d\lambda(a,\epsilon)$$

(Asset Market Clearing)

$$\int_{X\times E} c^*(a,\epsilon;\lambda)d\lambda(a,\epsilon) + \int_{X\times E} a^*(a,\epsilon;\lambda)d\lambda(a,\epsilon) = F(K^f(\lambda),\bar{H}) + (1-\delta)K^f(\lambda).$$

(Goods Market Clearing)

iv) **Belief Consistency**

The perceived law of motion is the actual law of motion: for any $\lambda \in \mathcal{M}_p(\mathcal{S})$, $\Psi(\lambda)$ is defined as the measure-kernel

$$\Psi(\lambda) = \lambda Q_\lambda := \int_S Q_\lambda(s, \cdot) d\lambda(s)$$

where Q_λ is the Markov kernel on $(S, \mathcal{S})$ defined as

$$Q_\lambda((a, \epsilon), B) = \int_S I_B(a^*(a, \epsilon; \lambda), \epsilon') P(\epsilon, d\epsilon')$$

for any $(a, \epsilon) \in S$ and $B \in \mathcal{S}$.

Of course, the RCE defined above can only remain a theoretical construct, since it is almost impossible to solve for an infinite-dimensional state variable (the distribution λ) via first order methods. To turn this into a computationally feasible problem, one simple option is to search for equilibria in which the macroeconomic aggregates stay constant over time, that is, where the law of motion for the distribution is given as $\lambda = \Psi(\lambda)$. At such an equilibrium, the household can perfectly predict prices today and tomorrow if it knows today's prices, so that the dependence on the distribution λ can be omitted and the model can be solved in the usual manner.

When the model admits aggregate uncertainty, this approach is infeasible, since the aggregate uncertainty means that macroeconomic aggregates cannot remain constant over time. In this case, we can rely on Krusell and Smith's approach, where they provide an approximate law of motion for macroeconomic aggregates, so that we no longer need to find the full law of motion $\Psi(\cdot)$ for the cross-sectional distributions.

7.2.3 Stationary Recursive Competitive Equilibrium

We now consider a stationary RCE, at which the cross-sectional distribution of wealth and productivity, and by extension macroeconomic aggregates and factor prices, remain constant over time. This indicates that, given a pair of fixed prices (w, r), the household's problem can be reformulated as

$$V(a, \epsilon; w, r) = \max_{(c, a') \in [0, +\infty) \times \mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a', \epsilon') P(\epsilon, d\epsilon') \right]$$

$$\text{subject to} \quad c + a' = w\epsilon + (1+r)a,$$

$$a' \geq -\phi.$$

The dependence of the value function on the distribution λ and its perceived law of motion can be replaced by a dependence on today's prices (w, r) because knowing today's prices is sufficient for knowledge of tomorrow's prices when the cross-sectional distribution of wealth and productivity is constant across time. In this sense, the pair of factor prices becomes the aggregate state variable, which reduces the dimension of the aggregate state from infinity to 2, which is why the problem becomes tractable.

Formally, a stationary RCE is defined as a collection of:

- A value function $V^* : S \times \mathbb{R}^2 \to \mathbb{R}$
- Policy functions $c^* : S \times \mathbb{R}^2 \to \mathbb{R}_+$ and $a^* : S \times \mathbb{R}^2 \to X$ for consumption and savings
- The firm's capital demand function $K^f : \mathbb{R}^2 \to \mathbb{R}_+$
- Prices $(w^*, r^*) \in \mathbb{R}^2$
- A distribution λ on $(S, \mathcal{S})$

that satisfy the following conditions:

i) **Household Optimality**
Each household optimizes given any pair of prices (w, r). Specificcally, V^* solves the functional equation

$$V(a, \epsilon; w, r) = \max_{(c,a') \in [0,+\infty) \times \mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a', \epsilon') P(\epsilon, d\epsilon') \right]$$

$$\text{subject to} \quad c + a' = w\epsilon + (1+r)a,$$

$$a' \geq -\phi.$$

$a^* : S \times \mathbb{R}^2 \to X$ is the associated savings policy function, and $c^* : S \times \mathbb{R}^2 \to \mathbb{R}_+$ is defined as

$$c^*(a, \epsilon; w, r) = w\epsilon + (1+r)a - a^*(a, \epsilon; w, r)$$

for any $(a,\epsilon) \in S$ and $(w,r) \in \mathbb{R}^2$.

ii) **Profit Maximization**

For any pair of prices (w,r), the firm chooses $K^f(w,r)$ in a manner that satisfies the first order conditions of profit maximizaiton:

$$w = F_h(K^f(w,r), \bar{H}), \quad r + \delta = F_k(K^f(w,r), \bar{H}).$$

iii) **Market Clearing**

All markets clear under the prices (w^*, r^*):

$$\bar{H} = \int_{X \times E} \epsilon d\lambda(a,\epsilon) \qquad \text{(Labor Market Clearing)}$$

$$K^f(w^*, r^*) = \int_{X \times E} a d\lambda(a,\epsilon) \qquad \text{(Asset Market Clearing)}$$

$$\int_{X \times E} c^*(a,\epsilon; w^*, r^*) d\lambda(a,\epsilon) + \int_{X \times E} a^*(a,\epsilon; w^*, r^*) d\lambda(a,\epsilon) = F(K^f(w^*, r^*), \bar{H}) + (1-\delta) K^f(w^*, r^*).$$

(Goods Market Clearing)

iv) **Stationarity of Cross-Sectional Distribution**

The distribution λ is unchanged over time given the prices (w^*, r^*):

$$\lambda = \int_S Q_{w^*,r^*}(s, \cdot) d\lambda(s)$$

where $Q_{w,r}$ is the Markov kernel on $(S, \mathcal{S})$ defined as

$$Q_{w,r}((a,\epsilon), B) = \int_S I_B(a^*(a,\epsilon; w, r), \epsilon') P(\epsilon, d\epsilon')$$

for any $(a,\epsilon) \in S$, $B \in \mathcal{S}$ and $(w,r) \in \mathbb{R}^2$.

7.2.4 The Firm's Static Problem and the Wage Function

A further simplification to stationary RCE can be made by utilizing the profit maximization condition. Let $K^f : \mathbb{R}^2 \to \mathbb{R}_+$ be a capital demand function that satisfies the first order conditions for profit maximization; for any $(w,r) \in \mathbb{R}^2$,

$$w = F_h(K^f(w,r), \bar{H}), \quad r+\delta = F_k(K^f(w,r), \bar{H}).$$

Euler's theorem for homogeneous functions of degree 0 now tells us that

$$F(K^f(w,r), \bar{H}) = F_h(K^f(w,r), \bar{H})\bar{H} + F_k(K^f(w,r), \bar{H})K^f(w,r) = w\bar{H} + (r+\delta)K^f(w,r),$$

so that the wage is a function of capital demand $K^f(w,r)$ and the interest rate r:

$$w = \frac{1}{\bar{H}}\left[F(K^f(w,r), \bar{H}) - (r+\delta)K^f(w,r)\right] = f\left(\frac{K^f(w,r)}{\bar{H}}\right) - (r+\delta)\frac{K^f(w,r)}{\bar{H}}.$$

Returning to the condition

$$r+\delta = F_k(K^f(w,r), \bar{H}),$$

the fact that the partial derivatives of functions that are homogeneous of degree one are homogeneous of degree 0 indicates that

$$r+\delta = F_k\left(\frac{K^f(w,r)}{\bar{H}}, 1\right) = f_k\left(\frac{K^f(w,r)}{\bar{H}}\right).$$

Therefore,

$$\frac{K^f(w,r)}{\bar{H}} = f_k^{-1}(r+\delta),$$

and we can finally see that

$$w = f(f_k^{-1}(r+\delta)) - (r+\delta)f_k^{-1}(r+\delta).$$

Since $f(\cdot)$ and $f_k^{-1}(\cdot)$ are all continuous functions, w is a continuous function of r; we can write this function as $w(\cdot)$.

To investigate further the properties of the wage function, note that

$$w(r) = F_h\left(\frac{K^f(w,r)}{\bar{H}}, 1\right) = F_h\left(f_k^{-1}(r+\delta), 1\right).$$

Here, $f_k^{-1}(\cdot)$ is strictly decreasing on $(0, +\infty)$ and $F_h(\cdot, 1)$ is strictly increasing on $[0, +\infty)$ with $F_h(0,1) = 0$ and $F_h(k,1) \to +\infty$ as $k \to \infty$, so that $w(\cdot)$ is strictly decreasing on $(-\delta, +\infty)$ with

$$\lim_{r\to\infty} w(r) = 0, \quad \lim_{r\downarrow -\delta} w(r) = +\infty.$$

These serve as the Inada conditions for the wage function.

The Simplified Stationary RCE

Recall from the income fluctuation problem that the household's asset space is unbounded if $\beta(1+r) \geq 1$. This observation, in conjunction with the properties of the wage function we have just obtained, indicates that we will be interested in equilibrium interest rates in the set

$$R = \left(-\delta, \frac{1}{\beta} - 1\right).$$

As such, we can narrow our search for stationary RCE to a search of:

- A value function $V^* : S \times R \to \mathbb{R}$
- Policy functions $c^* : S \times R \to \mathbb{R}_+$ and $a^* : S \times R \to X$ for consumption and savings
- The firm's capital demand function $K^f : R \to \mathbb{R}_+$
- An interest rate $r^* \in R$
- A distribution λ on $(S, \mathcal{S})$

that satisfy the following conditions:

i) **Household Optimality**
Each household optimizes given any interest rate $r \in R$. Specificcally, V^* solves

the functional equation

$$V(a,\epsilon;r) = \max_{(c,a')\in[0,+\infty)\times\mathbb{R}} \left[u(c) + \beta \cdot \int_E V(a',\epsilon')P(\epsilon,d\epsilon')\right]$$

$$\text{subject to} \quad c + a' = w(r)\epsilon + (1+r)a,$$

$$a' \geq -\phi.$$

$a^* : S \times R \to X$ is the associated savings policy function, and $c^* : S \times R \to \mathbb{R}_+$ is defined as

$$c^*(a,\epsilon;r) = w(r)\epsilon + (1+r)a - a^*(a,\epsilon;r)$$

for any $(a,\epsilon) \in S$ and $r \in R$.

ii) **Profit Maximization**

For any interest rate $r \in R$, the firm chooses $K^f(r)$ as

$$K^f(r) = \bar{H} \cdot f_k^{-1}(r+\delta).$$

iii) **Market Clearing**

All markets clear under the equilibrium interest rate r^*:

$$\bar{H} = \int_{X\times E} \epsilon d\lambda(a,\epsilon) \qquad \text{(Labor Market Clearing)}$$

$$K^f(r^*) = \int_{X\times E} a d\lambda(a,\epsilon) \qquad \text{(Asset Market Clearing)}$$

$$\int_{X\times E} c^*(a,\epsilon;r^*)d\lambda(a,\epsilon) + \int_{X\times E} a^*(a,\epsilon;r^*)d\lambda(a,\epsilon) = F(K^f(r^*),\bar{H}) + (1-\delta)K^f(r^*).$$

(Goods Market Clearing)

iv) **Stationarity of Cross-Sectional Distribution**

The distribution λ is unchanged over time given the equilibrium interest rate r^*:

$$\lambda = \int_S Q_{r^*}(s, \cdot) d\lambda(s)$$

where Q_r is the Markov kernel on $(S, \mathcal{S})$ defined as

$$Q_r((a, \epsilon), B) = \int_S I_B(a^*(a, \epsilon; r), \epsilon') P(\epsilon, d\epsilon')$$

for any $(a, \epsilon) \in S$, $B \in \mathcal{S}$ and $r \in R$.

Establishing existence of a stationary RCE in this economy is not a trivial task, since, unlike in the stochastic neoclassical growth model, the asset market clearing condition is not trivial. The method of proving existence is similar to how it usually goes when proving the existence of competitive equilibrium for a static economy. First, we note that the labor market clearing condition holds trivially due to inelastic labor supply and the assumption that the unconditional distribution of productivity is simply the stationary distribution of P. Therefore, by Walras' law, existence will be proved if we can show that either the asset market or the goods market clears. We choose to show that there exists a pair of prices that equilibrates the asset market by utilizing the result on the continuity of stationary distributions of parameter-dependent Markov chains.

7.2.5 Existence of Stationary RCE

In order for stationary RCE of the above form to exist, we first need to make the following assumption about the borrowing limit:

Borrowing Limts Define $\underline{w} = w\left(\frac{1}{\beta} - 1\right) > 0$. Then, for any $r \in (-\delta, 1/\beta - 1)$, $w(r) > \underline{w}$. We assume that

$$0 \leq \phi \leq \frac{\underline{w}\epsilon}{\frac{1}{\beta} - 1}.$$

The restriction that $\phi \geq 0$ ensures that households can freely lend, while the second inequality ensures that

$$r\phi - w(r)\underline{\epsilon} < \left(\frac{1}{\beta} - 1\right)\phi - \underline{w}\epsilon \leq 0$$

for any $r \in R$, which is the condition required for the household's constraint correspondence given the interest rate r to be non-empty valued.

We first show that the usual sufficient conditions under which the household's problem is well-defined hold for the Aiyagari model as well:

B1 : **Assupmtions on State and Productivity Spaces**
X and E are Borel subspaces of euclidean spaces, so that they are complete and separable metric spaces.

B2 : **Feller Property and Monotonicity of Productivity Shocks**
The Markov kernel P associated with the productivity process is assumed to possess the Feller and monotonicity properties.

B3 : **Well-behaved Constraint Correspondence**
Define the correspondence $\Gamma : X \times E \times R \to X$ as

$$\Gamma(a, \epsilon; r) = [-\phi,\ w(r)\epsilon + (1+r)a]$$

for any $(a, \epsilon, r) \in X \times E \times R$. $\Gamma(a, \epsilon; r)$ is non-empty because

$$w(r)\epsilon + (1+r)a \geq w(r)\epsilon - (1+r)\phi \geq \underline{w\epsilon} - \frac{1}{\beta}\phi > -\phi$$

by the restrictions imposed on ϕ.

The mapping $(a, \epsilon, r) \mapsto w(r)\epsilon + (1+r)a$ is a continuous function, and $\Gamma(\cdot)$ is non-empty compact and convex valued, so it follows that Γ is a non-empty compact and convex valued continuous correspondence.

Define the graph of Γ by

$$\mathrm{Gr}(\Gamma) = \{(a, a', \epsilon, r) \mid a' \in \Gamma(a, \epsilon; r)\}.$$

By the non-empty compact-valuedness and upper hemicontinuity of Γ coupled with the closed graph theorem, we can see that $\mathrm{Gr}(\Gamma)$ is a closed subset of $X \times X \times E \times R$.

B4 : **Well-behaved Instantaneous Return Function**

The return function $U : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ corresponding to the household's problem can be written as

$$U(a, a', \epsilon; r) = u\left[w(r)\epsilon + (1+r)a - a'\right]$$

for any $(a, a', \epsilon, r) \in \mathrm{Gr}(\Gamma)$. $U(\cdot)$ is a composition of continuous functions, so it is continuous on $\mathrm{Gr}(\Gamma)$. Furthermore, $u(\cdot)$ is a bounded function by assumption, so $U(\cdot)$ is also bounded on its domain.

B5 : **Monotonicity Assumptions**

Fixing $r \in R$, for any $(a', \epsilon) \in X \times E$ we want to show that $U(\cdot, a', \epsilon; r)$ is strictly increasing on X. This follows easily because $u(\cdot)$ is a strictly increasing function and the mapping

$$a \mapsto w(r)\epsilon + (1+r)a - a'$$

is strictly increasing in a as well. Furthermore, for any $\epsilon \in E$, since

$$w(r)\epsilon + (1+r)a_1 < w(r)\epsilon + (1+r)a_2$$

for any $a_1, a_2 \in X$ such that $a_1 < a_2$, we can see that

$$\Gamma(a_1, \epsilon; r) \subset \Gamma(a_2, \epsilon; r).$$

Furthermore, for any fixed $a \in X$, $U(a, \cdot; r)$ is strictly increasing on E because $u(\cdot)$ is strictly increasing and the mapping

$$\epsilon \mapsto w(r)\epsilon + (1+r)a - a'$$

is also strictly increasing ($w(r) > \underline{w} > 0$). We also already assumed that P is monotonic.

B6 : **Concavity Assumptions**

Fix $r \in R$. Choose any $\lambda \in (0,1)$ and $(a,\epsilon) \in X \times E$ and distinct $(a',a'') \in X$; then, because $u(\cdot)$ is a strictly concave function and

$$w(r)\epsilon + (1+r)a - \left(\lambda a' + (1-\lambda)a''\right) = \lambda\left(w(r)\epsilon + (1+r)a - a'\right) + (1-\lambda)\left(w(r)\epsilon + (1+r)a - a''\right),$$

where $w(r)\epsilon + (1+r)a - a' \neq w(r)\epsilon + (1+r)a - a''$, we have

$$\begin{aligned} U(a, \lambda a' + (1-\lambda)a'', \epsilon; r) &> \lambda \cdot u\left[w(r)\epsilon + (1+r)a - a'\right] + (1-\lambda)u\left[w(r)\epsilon + (1+r)a - a''\right] \\ &= \lambda U(a,a',\epsilon;r) + (1-\lambda)U(a,a'',\epsilon;r). \end{aligned}$$

This proves that the mapping

$$a' \mapsto U(a,a',\epsilon;r)$$

is strictly concave on X.

Now choose any $\epsilon \in E$ and pairs $(a_1,a_1'),(a_2,a_2') \in X \times X$. Then, for any $\lambda \in [0,1]$,

$$\begin{aligned} w(r)\epsilon + (1+r)\left(\lambda a_1 + (1-\lambda)a_2\right) - \left(\lambda a_1' + (1-\lambda)a_2'\right) \\ = \lambda\left[w(r)\epsilon + (1+r)a_1 - a_1'\right] + (1-\lambda)\left[w(r)\epsilon + (1+r)a_2 - a_2'\right], \end{aligned}$$

and the concavity of $u(\cdot)$ implies that

$$U\left[\lambda(a_1,a_1') + (1-\lambda)(a_2,a_2'),\epsilon;r\right] \geq \lambda U(a_1,a_1',\epsilon;r) + (1-\lambda)U(a_2,a_2',\epsilon;r).$$

Therefore, $U(\cdot,\epsilon;r)$ is concave.

To show the convexity of the constraint correspondence, choose any $\epsilon \in E$ and

$a, a' \in X$. Then, for any $\lambda \in [0,1]$ and $x \in \Gamma(a,\epsilon)$, $x' \in \Gamma(a',\epsilon)$, we have

$$-\phi \leq x \leq w(r)\epsilon + (1+r)a$$

$$-\phi \leq x' \leq w(r)\epsilon + (1+r)a',$$

so that

$$-\phi \leq \lambda x + (1-\lambda)x' \leq w(r)\epsilon + (1+r)\left(\lambda a + (1-\lambda)a'\right).$$

Since $\lambda x + (1-\lambda)x' \in X$ by the convexity of X, this implies that

$$\lambda x + (1-\lambda)x' \in \Gamma(\lambda a + (1-\lambda)a', \epsilon),$$

so that $\Gamma(\cdot)$ is appropriately convex.

B7 : **Differentiability Assumptions**

X has a non-empty interior $(-\phi, +\infty)$, and for any $r \in R$ and $\epsilon \in E$, the mapping

$$(a,a') \mapsto U(a,a',\epsilon;r) = u\left[w(r)\epsilon + (1+r)a - a'\right]$$

is continuously differentiable.

Due to the fact that the value function also explicitly depends on the parameter r as well as the states (a,ϵ), we must re-derive the general results to allow for the savings and consumption policy functions, as well as the value function, to depend continuously in r.

To this end, define the operator $T : C_b(S \times R, \mathbb{R}) \to C_b(S \times R, \mathbb{R})$ as

$$[T(f)](a,\epsilon;r) = \sup_{a' \in \Gamma(a,\epsilon;r)} \left[U(a,a',\epsilon;r) + \beta \int_E f(a',\epsilon';r) P(\epsilon, d\epsilon')\right].$$

for any $f \in C_b(S \times R, \mathbb{R})$ and $(a,\epsilon,r) \in S \times R$. Note that the mapping

$$(a,a',\epsilon,r) \mapsto U(a,a',\epsilon;r) + \beta \int_E f(a',\epsilon';r) P(\epsilon, d\epsilon')$$

is continuous and bounded on $\mathrm{Gr}(\Gamma)$ by the generalized Feller property (for the continuity of the integral term) and the assumed continuity of $U(\cdot)$. Since $\Gamma(\cdot)$ is a non-empty compact convex valued correspondence that is also continuous, Berge's maximum theorem now tells us that the value function $T(f)$ corresponding to the problem

$$\max_{a' \in \Gamma(a,\epsilon;r)} \quad \left[U(a,a',\epsilon;r) + \beta \int_E f(a',\epsilon';r) P(\epsilon, d\epsilon') \right]$$

is real-valued and continuous on $S \times R$. $T(f)$ is also bounded because this maximum is attained, and the objective function is bounded. Thus, the operator T does indeed map $C_b(S \times R, \mathbb{R})$ into itself. Now we can show that T is a contraction mapping on the complete metric space $(C_b(S \times R, \mathbb{R}), d_\infty)$, where d_∞ is the supremum metric on $C_b(S \times R, \mathbb{R})$, using Blackwell's sufficient conditions:

Monotonicity For any $f, g \in C_b(S \times R, \mathbb{R})$ such that $f \leq g$,

$$U(a,a',\epsilon;r) + \beta \int_E f(a',\epsilon';r) P(\epsilon, d\epsilon') \leq U(a,a',\epsilon;r) + \beta \int_E g(a',\epsilon';r) P(\epsilon, d\epsilon')$$

for any $(a,a',\epsilon,r) \in \mathrm{Gr}(\Gamma)$ by the monotonicity of integration, so that $T(f) \leq T(g)$ on $S \times R$.

Discounting For any $f \in C_b(S \times R, \mathbb{R})$ and $c \geq 0$,

$$\begin{aligned} U(a,a',\epsilon;r) + \beta \int_E (f+c)(a',\epsilon';r) P(\epsilon, d\epsilon') & \\ = U(a,a',\epsilon;r) + \beta \int_E f(a',\epsilon';r) P(\epsilon, d\epsilon') + \beta c & \\ \leq [T(f)](a,\epsilon,r) + \beta c & \end{aligned}$$

for any $(a,a',\epsilon,r) \in \mathrm{Gr}(\Gamma)$, so that $T(f+c) \leq T(f) + \beta c$, where $\beta \in (0,1)$.

This shows that T is a contraction mapping on $(C_b(S \times R, \mathbb{R}), d_\infty)$, and by the Banach fixed point theorem there exists a unique $V^* \in C_b(S \times R, \mathbb{R})$ such that $T(V^*) = V^*$, or equivalently,

$$V^*(a,\epsilon;r) = \sup_{a' \in \Gamma(a,\epsilon;r)} \left[U(a,a',\epsilon;r) + \beta \int_E V^*(a',\epsilon';r) P(\epsilon, d\epsilon') \right].$$

Berge's maximum theorem now tells us that the maximizer correspondence $G : S \times R \to X$ defined as

$$G(a, \epsilon; r) = \left\{ a' \in \Gamma(a, \epsilon; r) \mid U(a, a', \epsilon; r) + \beta \int_E V^*(a', \epsilon'; r) P(\epsilon, d\epsilon') = V^*(a, \epsilon; r) \right\}$$

for any $(a, \epsilon, r) \in S \times R$ is non-empty compact valued and upper hemicontinuous.

It now follows from our monotonicity and concavity assumptions and the usual application of the properties theorem for each $r \in R$ that:

- $V^*(\cdot, \epsilon; r)$ is strictly increasing on X for any $\epsilon \in E$
- $V^*(a, \cdot; r)$ is strictly increasing on E for any $a \in X$
- $V^*(\cdot, \epsilon; r)$ is strictly concave on X for any $\epsilon \in E$
- $G(\cdot; r)$ is a single-valued correspondence.

We can therefore define $a^* : S \times R \to X$ as the selection

$$a^*(a, \epsilon; r) \in G(a, \epsilon; r)$$

for any $(a, \epsilon, r) \in S \times R$. Since $G(\cdot)$ is a single-valued upper hemicontinuous correspondence, it follows that $a^*(\cdot)$ is continuous on $S \times R$. The corresponding consumption policy function $c^* : S \times R \to \mathbb{R}_+$ is defined as

$$c^*(a, \epsilon; r) = w(r)\epsilon + (1 + r)a - a^*(a, \epsilon; r).$$

Since the wage function $w(\cdot)$ and the savings policy function $a^*(\cdot)$ are both continuous, $c^*(\cdot)$ is also continuous on $S \times R$.

For each $r \in R$, the household faces the same income fluctuation problem as in the previous section. Therefore, the same results derived from the income fluctuation problem apply in the current setting as well. We enumerate them below for the sake of completeness:

1. **Positivity of Consumption**
   For any $(a, \epsilon) \in S$, we have $c^*(a, \epsilon; r) > 0$.

2. **Envelope Theorem for Value Function**
   For any $\epsilon \in E$, $V^*(\cdot, \epsilon; r)$ is differentiable on $(-\phi, +\infty)$ with derivative given as

$$\nabla_a V^*(a, \epsilon; r) = u'[c^*(a, \epsilon; r)](1 + r).$$

   Furthermore, for any $(a, \epsilon) \in S$ such that $a^*(a, \epsilon; r) > -\phi$,

$$\nabla_a V^*(a^*(a, \epsilon; r), \epsilon'; r) = u'\left[c^*(a^*(a, \epsilon; r), \epsilon'; r)\right](1 + r)$$

   for any $\epsilon' \in E$.

3. **An Envelope Result for the Value Function at the Boundary**
   For any $\epsilon \in E$, there exists some $\nabla_{a+} V^*(-\phi, \epsilon; r) \in \mathbb{R}$ such that

$$\lim_{h \downarrow 0} \frac{V^*(-\phi + h, \epsilon; r) - V^*(-\phi, \epsilon; r)}{h} = \nabla_{a+} V^*(-\phi, \epsilon; r),$$

   and

$$\nabla_{a+} V^*(-\phi, \epsilon; r) \geq u'[c^*(-\phi, \epsilon; r)](1 + r).$$

4. **The Euler Inequality**
   For any $(a, \epsilon) \in S$,

$$u'[c^*(a, \epsilon; r)] \geq \beta(1 + r) \int_E u'\left[c^*(a^*(a, \epsilon; r), \epsilon'; r)\right] P(\epsilon, d\epsilon'),$$

   with the inequality holding as an equality if $a^*(a, \epsilon; r) > -\phi$.

5. **Monotonicity of Policy Functions**

   For any $\epsilon \in E$, $c^*(\cdot, \epsilon; r)$ is strictly increasing on X and $a^*(\cdot, \epsilon; r)$ is increasing on X.

   If the endowments are drawn iid from some distribution v on $(E, \mathcal{E})$, then for any $a \in X$, $c^*(a, \cdot; r)$ is strictly increasing on E and $a^*(a, \cdot; r)$ is increasing on E.

6. **Unboundendness of Consumption**

   For any $\epsilon \in E$,

$$\lim_{a \to \infty} c^*(a, \epsilon; r) = +\infty.$$

7. **Boundedness of Asset Space**

   Since $\beta(1 + r) < 1$, if the endowment shocks are iid and $u'(\cdot)$ is twice continuous differentiable and convex, then there exists some $\bar{a}(r) > 0$ such that $a^*(a, \epsilon; r) < a$ for any $a \geq \bar{a}(r)$ and $\epsilon \in E$.

8. **Existence of Unique Stationary Distribution**

   Define the Markov kernel Q_r on $(S, \mathcal{S})$ as

$$Q_r((a, \epsilon), B) = \int_E I_B(a^*(a, \epsilon; r), \epsilon') P(\epsilon, d\epsilon')$$

   for any $B \in \mathcal{S}$. Then, Q_r possesses the Feller property.

   If the endowment shocks are iid, $u'(\cdot)$ is a twice continuously differentiable convex function, and $a^*(-\phi, \overline{\epsilon}; r) > -\phi$, then Q_r is a monotonic kernel and there exists a unique stationary distribution λ_r on $(S, \mathcal{S})$ such that

$$\lambda_r([-\phi, \overline{a}_m(r)] \times E) = 1,$$

where $\overline{a}_m(r)$ is defined as

$$\overline{a}_m(r) = \inf\{a \in X \mid a^*(a, \overline{\epsilon}; r) = a\} < +\infty.$$

Since $a^*(\cdot)$ is continuous on $S \times R$, for any $\{(s_n, r_n)\}_{n \in N_+}$ in $S \times R$ that converges to some $(s, r) \in S \times R$, we have

$$Q_{r_n}(s_n, \cdot) \to Q_r(s, \cdot)$$

weakly as $n \to \infty$. To apply the continuity theorem, it remains to show that, for any convergent sequence $\{r_n\}_{n \in N_+}$ in R, there exists a single compact set on which the measures $\{\lambda_r\} \cup \{\lambda_{r_n}\}_{n \in N_+}$ are concentrated. This can be shown by applying a uniform version of the boundedness theorem that we proved for the income fluctuation problem.

Theorem (Uniform Boundedness of Asset Spaces)
Consider the Aiyagari model above, and make the following additional assumptions:

i) The endowment shocks are iid and drawn from a distribution v on $(E, \mathcal{E})$ such that $v(\{\underline{\epsilon}\}) > 0$ and $v(\{\overline{\epsilon}\}) > 0$.

ii) $u'(\cdot)$ is a twice continuously differentiable convex function.

iii) $a^*(-\phi, \overline{\epsilon}; r) > -\phi$ for any $r \in R$.

Then, for any compact subset R_0 of R, there exists an $\overline{a} > 0$ such that

$$\lambda_r([-\phi, \overline{a}] \times E) = 1$$

for any $r \in R_0$.

Proof) We will first show that there exists some $A > 0$ such that $a^*(a, \overline{\epsilon}; r) < a$ for any $a \geq A$ and $r \in R_0$.

By assumption, $a^*(-\phi, \overline{\epsilon}; r) > -\phi$ and $a^*(\cdot; r)$ is monotonically increasing on S

for any $r \in R$, so the following Euler equation holds for any $a \in X$ and $r \in R$:

$$u'\left[c^*(a,\overline{\epsilon};r)\right] = \beta(1+r)\int_E u'\left[c^*(a^*(a,\overline{\epsilon};r),\epsilon';r)\right]dv(\epsilon')$$

$$= \beta(1+r)\cdot\frac{\int_E u'\left[c^*(a^*(a,\overline{\epsilon};r),\epsilon';r)\right]dv(\epsilon')}{u'\left[c^*(a^*(a,\overline{\epsilon};r),\overline{\epsilon};r)\right]}u'\left[c^*(a^*(a,\overline{\epsilon};r),\overline{\epsilon};r)\right].$$

Because $c^*(\cdot;r)$ is strictly increasing in both arguments,

$$1 \leq \frac{\int_E u'\left[c^*(a^*(a,\overline{\epsilon};r),\epsilon';r)\right]dv(\epsilon')}{u'\left[c^*(a^*(a,\overline{\epsilon};r),\overline{\epsilon};r)\right]} \leq \frac{u'\left[c^*(a^*(a,\overline{\epsilon};r),\underline{\epsilon};r)\right]}{u'\left[c^*(a^*(a,\overline{\epsilon};r),\overline{\epsilon};r)\right]}.$$

The same arguments as in the proof of the boundedness theorem in the previous section indicates that the right hand side can be bounded as follows:

$$1 \geq \frac{u'\left[c^*(a^*(a,\overline{\epsilon};r),\overline{\epsilon};r)\right]}{u'\left[c^*(a^*(a,\overline{\epsilon};r),\underline{\epsilon};r)\right]} \geq 1-\mu\frac{\overline{\epsilon}-\underline{\epsilon}}{c^*(a^*(a,\overline{\epsilon};r),\underline{\epsilon};r)}.$$

Now suppose that there does not exist any $A > 0$ such that $a^*(a,\overline{\epsilon};r) < a$ for any $a \geq A$ and $r \in R_0$. This means that no positive number can serve the role of A, so that we can construct a sequence $\{a_n, r_n\}_{n\in N_+}$ in $X \times R_0$ such that $\{a_n\}_{n\in N_+}$ is strictly increasing and

$$a^*(a_n,\overline{\epsilon};r_n) > a_n$$

for any $n \in N_+$[3]. Since $\{r_n\}_{n\in N_+}$ is a sequence in the compact set R_0, by the equivalence of compactness and sequential compactness in metric spaces there exists a subsequence of $\{r_n\}_{n\in N_+}$ that converges to some $r^L \in R_0$; for notational simplicity, let $\{r_n\}_{n\in N_+}$ itself be this subsequence.

Denote by $r^* = \sup R_0$; $r^* \in R_0$ because R_0 is closed and bounded (Heine-Borel). Since $\beta(1+r^*) < 1$, $c^*(a,\overline{\epsilon};r^L) \to +\infty$ as $a \to \infty$ and $a_n \to +\infty$ as $n \to \infty$ by construction, it follows that there exists some $N \in N_+$ such that

$$c^*(a_N,\overline{\epsilon};r^L) > 1 + \frac{\mu}{1-\beta(1+r^*)},$$

[3]We can construct a_n by choosing $a_1 > 1$ and, for any $n \geq 1$, choosing $a_{n+1} > a_n$ in a manner that satisfies the above condition.

or equivalently,

$$\beta(1+r^*) < 1 - \mu \frac{1}{c^*(a_N, \overline{\epsilon}; r^L) - 1}.$$

The mapping

$$r \mapsto c^*(a_N, \overline{\epsilon}; r)$$

is continuous on R and $r_n \to r^L$ as $n \to \infty$, so there exists some $N_1 > N$ such that

$$c^*(a_N, \overline{\epsilon}; r^L) - 1 < c^*(a_N, \overline{\epsilon}; r_n)$$

for any $n \geq N_1$. $\{a_n\}_{n \in N_+}$ is a strictly increasing sequence, $c^*(\cdot, \overline{\epsilon}; r_n)$ is strictly increasing on X for any n, and $a_n < a^*(a_n, \overline{\epsilon}; r_n)$ for any n, so it follows that

$$c^*(a_N, \overline{\epsilon}; r^L) - 1 < c^*(a_n, \overline{\epsilon}; r_n) < c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)$$

for any $n \geq N_1$. This implies that, for any $n \geq N_1$,

$$\begin{aligned} \beta(1+r^*) < 1 - \mu \frac{1}{c^*(a_N, \overline{\epsilon}; r^L) - 1} &< 1 - \mu \frac{1}{c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)} \\ &\leq \frac{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right]}{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \underline{\epsilon}; r_n)\right]}. \end{aligned}$$

Therefore,

$$1 \leq \frac{\int_E u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \epsilon'; r_n)\right] dv(\epsilon')}{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right]} \leq \left(\frac{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right]}{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \underline{\epsilon}; r_n)\right]} \right)^{-1} < \frac{1}{\beta(1+r^*)}$$

and in particular

$$\begin{aligned} u'\left[c^*(a_n, \overline{\epsilon}; r_n)\right] &= \beta(1+r_n) \frac{\int_E u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \epsilon'; r_n)\right] dv(\epsilon')}{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right]} u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right] \\ &\leq \beta(1+r^*) \frac{\int_E u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \epsilon'; r_n)\right] dv(\epsilon')}{u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right]} u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right] \\ &< u'\left[c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)\right] \end{aligned}$$

for any $n \geq N_1$. By implication,

$$c^*(a_n, \overline{\epsilon}; r_n) > c^*(a^*(a_n, \overline{\epsilon}; r_n), \overline{\epsilon}; r_n)$$

and therefore, because $c^*(\cdot, \overline{\epsilon}; r_n)$ is strictly increasing on X for any n,

$$a_n > a^*(a_n, \overline{\epsilon}; r_n)$$

for any $n \geq N_1$. This contradicts the way we constructed $\{a_n, r_n\}_{n \in N_+}$, so there must exist some $A > 0$ such that $a^*(a, \overline{\epsilon}; r) < a$ for any $a \geq A$ and $r \in R_0$.

It is now a simple matter to establish the claim in the theorem. Fix $r \in R_0$, and let $\overline{a}_m(r)$ be the smallest fixed point of the function $a^*(\cdot, \overline{\epsilon}; r)$. It must then be the case that $\overline{a}_m(r) < A$, since otherwise

$$a^*(\overline{a}_m(r), \overline{\epsilon}; r) < \overline{a}_m(r),$$

which contradicts the fact that $\overline{a}_m(r)$ is a fixed point of $a^*(\cdot, \overline{\epsilon}; r)$. Since

$$\lambda_r([-\phi, \overline{a}_m(r)] \times E) = 1$$

and $[-\phi, \overline{a}_m(r)] \times E \subset [-\phi, A] \times E$, it must also be the case that

$$\lambda_r([-\phi, A] \times E) = 1.$$

Q.E.D.

The continuity of the asset supply function now follows immediately from the continuity result established in chapter 3:

Theorem (Continuity of Asset Supply)
Consider the Aiyagari model above, and make the following additional assumptions:

i) The endowment shocks are iid and drawn from a distribution v on $(E, \mathcal{E})$ such that $v(\{\underline{\epsilon}\}) > 0$ and $v(\{\overline{\epsilon}\}) > 0$.

ii) $u'(\cdot)$ is a twice continuously differentiable convex function.

iii) $a^*(-\phi, \overline{\epsilon}; r) > -\phi$ for any $r \in R$.

Let $A : R \to \mathbb{R}$ be the asset supply function defined as

$$A(r) = \int_{X \times E} a \, d\lambda_r(a, \epsilon)$$

for any $r \in R$. Then, $A(\cdot)$ is continuous on R with

$$A(r) \to +\infty \quad \text{as} \quad r \to \frac{1}{\beta} - 1.$$

Proof) We have seen that, for any sequence $\{(s_n, r_n)\}_{n \in N_+}$ in $S \times R$ that converges to some $(s, r) \in S \times R$, we have

$$Q_{r_n}(s_n, \cdot) \to Q_r(s, \cdot)$$

weakly as $n \to \infty$. Furthermore, each Q_r admits a unique stationary distribution λ_r concentrated on the compact set $[-\phi, \overline{a}_m(r)] \times E$, where $\overline{a}_m(r)$ is the smallest fixed point of the function $a^(\cdot, \overline{\epsilon}; r)$. Finally, for any sequence $\{r_n\}_{n \in N_+}$ in R converging to some $r_0 \in R$, since $R_0 = \{r_n\}_{n \in N_+} \cup \{r_0\}$ is a compact subset of R, the preceding theorem tells us that there exists some $\overline{a} > 0$ such that

$$\lambda_{r_n}([-\phi, \overline{a}] \times E) = 1$$

for any $n \in \mathbb{N}$. It then follows from the continuity theorem that

$$\lambda_{r_n} \to \lambda_{r_0}$$

weakly as $n \to \infty$. Defining $f : S \to \mathbb{R}$ as

$$f(a,\epsilon) = \begin{cases} a & \text{for } -\phi \le a \le \overline{a} \\ 2\overline{a} - a & \text{for } \overline{a} < a \le 2\overline{a} \\ 0 & \text{for } 2\overline{a} < a \end{cases},$$

we can see that f is a continuous and bounded function on S such that

$$A(r_n) = \int_{X \times E} a \, d\lambda_{r_n}(a,\epsilon) = \int_{X \times E} f(a) d\lambda_{r_n}(a,\epsilon)$$

for any $n \in \mathbb{N}$ because λ_{r_n} assigns mass 1 to $[-\phi, \overline{a}] \times E$. By the definition of weak convergence, we now have

$$\lim_{n \to \infty} A(r_n) = \lim_{n \to \infty} \int_{X \times E} f(a) d\lambda_{r_n}(a,\epsilon) = \int_{X \times E} f(a) d\lambda_{r_0}(a,\epsilon) = A(r_0).$$

This holds for any sequence $\{r_n\}_{n \in N_+}$ in R that converges to a point in R, so $A(\cdot)$ is continuous on R.

Q.E.D.

Chapter 8

Firm Dynamics

So far, we have focused on

8.1 The Hopenhayn Model of Firm Dynamics

8.1.1 Model Assumptions

The model assumptions are as follows:

Firms There are infinitely many homogeneous firms whose measure is normalized to 1. These firms produce a single final good using a single input, labor. The final good is traded in the goods market, and labor is demanded from the labor market, where consumer demand and labor supply are assumed to be exogenous functions of total quantity produced and the wage level.

Households Instead of fully specifying the household's preferences, we assume simply that there exists an exogenous demand functions $D : \mathbb{R}_+ \to \mathbb{R}_+$ such that the price p_t of the final good satisfies

$$D(p_t) = Q_t,$$

where Q_t is the total quantity produced at time t. We assume that labor supply is perfectly elastic at the wage $w > 0$, which we normalize to 1.

Technologies We assume that the representative firm faces the production function $F : H \times Z \to \mathbb{R}_+$ defined as

$$F(n, z) = z \cdot f(n)$$

for any $(n, z) \in H \times Z$, where z is the productivity of the firm, n is the labor hours used for production, and $H = [0, +\infty)$ with corresponding Borel σ-algebra $\mathcal{H}$.

The function $f(\cdot)$ is assumed to satisfy the usual conditions for a well-behaved

production function:

$$f'(\cdot) > 0, \quad f''(\cdot) < 0 \quad \text{on } (0, +\infty),$$

$$\lim_{n \downarrow 0} f'(n) = +\infty, \quad \lim_{n \to \infty} f'(n) = 0.$$

This indicates that $f'(\cdot)$ is a strictly decreasing function on $(0, +\infty)$ that maps onto $(0, +\infty)$. Therefore, it admits an inverse function $(f')^{-1} : (0, +\infty) \to (0, +\infty)$ that is also strictly decreasing on $(0, +\infty)$. We also assume that there exists an $\alpha > 1$ such that

$$\frac{f(n)}{nf'(n)} \geq \alpha$$

for any $n > 0$.

The productivity space Z is assumed to be a compact subset of $(0, +\infty)$ with corresponding Borel σ-algebra $\mathcal{Z}$. Productivity evolves as a Markov chain with kernel P, which we assume satisfies the Feller property and is monotonic, as usual. Furthermore, we assume that P admits a stationary distribution γ.

Each firm's net revenue $\pi(\cdot)$ is defined as

$$\pi(p, n, z) = pzf(n) - n - \phi,$$

where $\phi > 0$ is a fixed operating cost of production, in contrast to the variable cost of production n.

Entry and Exit Firms are allowed to freely enter and exit the market at any time. If a firm decides to enter the market, then its productivity starting next period is drawn from the stationary distribution γ of P. A propective entrant must pay a fixed cost $\kappa > 0$ for entry.

Each period, the incumbent firm determines how much labor hours to use for production, and whether to remain in the market in the next period, while a non-incumbent firm chooses whether to enter or exit the market. We let $X = \{0, 1\}$ represent the set of choices that an incumbent firm can make

about whether to exit the market.

Both types of firms discount future profits by $\beta \in (0,1)$.

8.1.2 Stationary Recursive Competitive Equilibrium

As in the Aiyagari model, we will be interested in stationary RCE, where economic aggregates stay constant over time. In our model, the only economic aggregate of interest is the price p of the final good.

The Incumbent Firm's Problem

We first formulate the problem of the incumbent firm. Given price p, a firm with current productivity z must choose how much labor hours n to use in production, and whether to continue producing tomorrow. Letting $V^e(p)$ be the value of a non-incumbent firm given time-invariant price p, it follows that the incumbent firm's problem can be written as

$$\begin{aligned} V(z;p) &= \sup_{(n,x)\in H\times\{0,1\}} \Bigg[\pi(p,n,z) \\ &\qquad + \beta\cdot(1-x)\int_Z V(z';p)P(z,dz') + \beta x V^e(p)\Bigg] \\ &= \sup_{n\in H} \pi(p,n,z) + \beta\cdot\max\left[\int_Z V(z';p)P(z,dz'),\, V^e(p)\right]. \end{aligned}$$

Since labor hours n only appears in the term concerning today's profits, we can solve the static problem

$$\max_{n\in[0,+\infty)} \quad \pi(p,n,z) = pzf(n) - n - \phi$$

to obtain optimal labor demand as a function of prices and productivity: letting $n^* : Z \times \mathbb{R}_{++} \to [0,+\infty)$ be the optimal labor demand policy function, the first order conditions for maximization tell us that

$$pzf'(n^*(z;p)) = 1,$$

which can be solved for $n^*(z;p)$ as

$$n^*(z;p) = (f')^{-1}\left(\frac{1}{pz}\right) > 0.$$

It follows that the value function of the incumbent firm can be written as

$$V(z;p) = \pi(p, n^*(z;p), z) + \beta \cdot \max\left[\int_Z V(z';p)P(z,dz'),\ V^e(p)\right].$$

We can also formulate the optimal entry decision policy function $\chi^* : Z \times \mathbb{R}_{++} \to \{0,1\}$ as

$$\chi(z;p,w) = \begin{cases} 1 & \text{if } \int_Z V(z';p)P(z,dz') \leq V^e(p) \\ 0 & \text{if } \int_Z V(z';p)P(z,dz') > V^e(p) \end{cases}.$$

The Non-Incumbent Firm's Problem

Moving onto the problem of the non-incumbent firm, a non-incumbent firm can either choose to continue to remain outside the market, or it could choose to enter the market, at which point it must pay an entry cost of κ today and receive a payoff tomorrow based on the productivity it draws from the stationary distribution γ of P. Therefore, the non-incumbent firm's dynamic problem given the time-invariant price p is

$$V^e(p) = \max\left[\beta \cdot V^e(p),\ -\kappa + \beta \int_Z V(z;p)d\gamma(z)\right].$$

To obtain a sharper characterization of $V^e(p)$, suppose that $-\kappa + \beta \int_Z V(z;p)d\gamma(z) \leq \beta V^e(p)$. In this case,

$$V^e(p) = \beta V^e(p)$$

and thus $V^e(p) = 0$ and therefore $-\kappa + \beta \int_Z V(z;p)d\gamma(z) \leq 0$. Now suppose that $-\kappa + \beta \int_Z V(z;p)d\gamma(z) \leq 0$. In this case, if

$$\beta V^e(p) < -\kappa + \beta \int_Z V(z;p)d\gamma(z),$$

then

$$V^e(p) = -\kappa + \beta \int_Z V(z;p) d\gamma(z) < 0,$$

which would imply $\beta V^e(p) > V^e(p) = -\kappa + \beta \int_Z V(z;p) d\gamma(z)$, a contradiction. Therefore, it must be the case that $\beta V^e(p) \geq -\kappa + \beta \int_Z V(z;p) d\gamma(z)$.

We have just shown that

$$-\kappa + \beta \int_Z V(z;p) d\gamma(z) \leq \beta V^e(p) \quad \text{if and only if} \quad -\kappa + \beta \int_Z V(z;p) d\gamma(z) \leq 0.$$

This allows us to reformulate the non-incumbent firm's value function as

$$V^e(p) = \max\left[-\kappa + \beta \int_Z V(z;p) d\gamma(z),\ 0\right] \geq 0.$$

Characterizing Equilibrium

Suppose λ is a finite measure on $(Z, \mathcal{Z})$ such that $\lambda(Z) = M$ is the total mass of incumbent firms in the economy, and $\frac{1}{M}\lambda$ is a probability measure on $(Z, \mathcal{Z})$ that describes how productivity levels are distributed across incumbent firms. The following market clearing conditions must be satisfied at equilibrium:

$$p = D\left(\int_Z z f(n^*(z;p)) d\lambda(z)\right), \quad 1 = W\left(\int_Z n^*(z;p) d\lambda(z)\right),$$

since $\int_Z z f(n^*(z;p)) d\lambda(z)$ is aggregate production and $\int_Z n^*(z;p) d\lambda(z)$ is aggregate labor demand. By Walras' law, it suffices for just one of these conditions to hold for both of them to hold.

In order for prices to remain constant across time, we therefore require the "distribution" λ of productivity levels for incumbent firms to remain constant across time. Given that the current distribution of productivity levels across incumbent firms is λ, tomorrow's distribution of productivity levels across incumbent firms is given as

$$\int_Z (1 - \chi^*(z;p,w)) P(z', \cdot) d\lambda(dz') + E\gamma(\cdot),$$

where $\int_Z (1 - \chi^*(z;p,w)) P(z', \cdot) d\lambda(dz')$ represents the contribution of firms that have chosen to remain in the market today, and $E\gamma(\cdot)$ that of firms that enter the economy

tomorrow. Here, E is the mass of entrants that choose to enter the market tomorrow.

If $V^e(p) > 0$, then $-\kappa + \beta \int_Z V(z;p) d\gamma(z) > 0$ and therefore $-\kappa + \beta \int_Z V(z;p) d\gamma(z) > \beta V^e(p)$. This indicates that all potential entrants have an incentive to enter the market. The resulting influx of firms raises $\lambda(Z)$ and drives down prices, which in turn drives $V(z;p)$ down for each $z \in Z$. As such, at equilibrium it must be the case that $V^e(p) = 0$.

Defining Stationary RCE

In light of all the observations made above, we can now define a stationary RCE in our model as a collection of:

- Value functions $V^* : Z \times \mathbb{R}_{++} \to \mathbb{R}$ and $V^e : \mathbb{R}_{++} \to \mathbb{R}$,
- Labor demand and entry decision policy functions $g^* = (n^*, \chi^*) : Z \times \mathbb{R}_{++} \to [0, +\infty) \times \{0, 1\}$,
- An equilibrium price $p^* > 0$
- A finite measure λ on $(Z, \mathcal{Z})$
- A mass of entrants E

that satisfy the following conditions:

i) **Incumbent Firm Optimality**
$n^* : Z \times \mathbb{R}_{++} \to [0, +\infty)$ and $\chi^* : Z \times \mathbb{R}_{++} \to \{0, 1\}$ are optimal labor demand and market exit decision functions such that

$$\pi(p, n^*(z;p), z) = \sup_{n \geq 0} \pi(p, n, z)$$

and

$$\chi^*(z;p) = \begin{cases} 1 & \text{if } \int_Z V(z';p) P(z, dz') < V^e(p) \\ 0 & \text{if } \int_Z V(z';p) P(z, dz') > V^e(p) \end{cases}.$$

for any $z \in Z$ and $p > 0$.

Furthermore, V^* satisfies the following functional equation: for any $z \in Z$ and $p > 0$,

$$V^*(z;p) = \pi(p, n^*(z;p), z) + \beta \cdot \max\left[\int_Z V^*(z';p)P(z,dz'),\ V^e(p)\right].$$

ii) **Entrant Firm Optimality**

V^e satisfies the following functional equation:

$$V^e(p) = \max\left[-\kappa + \beta \int_Z V^*(z;p)d\gamma(z),\ 0\right]$$

for any $p > 0$.

iii) **Free Entry**

The value function of the non-incumbent firms are 0 at the optimal prices:

$$V^e(p^*) = 0 = -\kappa + \beta \int_Z V^*(z;p^*)d\gamma(z).$$

iv) **Market Clearing**

The goods market clears at the optimal prices:

$$D(p^*) = \int_Z z f(n^*(z;p^*))d\lambda(z).$$

v) **Stationary Distribution of Productivity**

The distribution of productivity across incumbent firms remains unchanged across time:

$$\lambda(\cdot) = \int_Z (1 - \chi^*(z;p^*))\, P(z,\cdot)d\lambda(z) + E \cdot \gamma(\cdot).$$

8.1.3 Existence of a Stationary Equilibrium Price

We first focus on the incumbent firm's recursive problem: we want to solve the functional equation

$$V(z;p) = \underbrace{\pi(p, n^*(z;p), z)}_{\pi^*(z;p)} + \beta \cdot \max\left[\int_Z V(z';p)P(z,dz'), 0\right].$$

for any $p > 0$ and productivity $z \in Z$. We first establish some results about the flow profit function $\pi^* : Z \times \mathbb{R}_{++} \to \mathbb{R}$.

Lemma (Monotonicity of Flow Profit Function)
Suppose that our model assumptions hold, and let $\pi^* : Z \times \mathbb{R}_{++} \to \mathbb{R}$ be the flow profit function be defined as above. Then, π^* is a continuous function on $Z \times \mathbb{R}_{++}$ that is strictly increasing in both productivity and prices. Furthermore, for any $z \in Z$ we have

$$\lim_{p \downarrow 0} \pi^*(z;p) = 0, \quad \lim_{p \to \infty} \pi^*(z;p) = +\infty.$$

Proof) By definition,

$$\pi^*(z;p) = pzf\left[(f')^{-1}\left(\frac{1}{pz}\right)\right] - (f')^{-1}\left(\frac{1}{pz}\right) - \phi$$

for any $z \in Z$ and $p > 0$. By the continuity of $f(\cdot)$ and $(f')^{-1}(\cdot)$, it follows that $\pi^*(\cdot)$ is continuous on $Z \times \mathbb{R}_{++}$.

Now note that

$$\begin{pmatrix} \frac{n^*(z;p)}{\partial z} \\ \frac{n^*(z;p)}{\partial p} \end{pmatrix} = \frac{1}{f''\left[(f')^{-1}\left(\frac{1}{pz}\right)\right]} \begin{pmatrix} -\frac{1}{pz^2} \\ -\frac{1}{p^2 z} \end{pmatrix}$$

for any $(z,p) \in Z^o \times (0, +\infty)$. It follows from the envelope theorem that

$$\begin{pmatrix} \frac{\partial \pi^*(z;p)}{\partial z} \\ \frac{\partial \pi^*(z;p)}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial \pi(p, n^*(z;p), z)}{\partial z} \\ \frac{\partial \pi(p, n^*(z;p), z)}{\partial p} \end{pmatrix} = \begin{pmatrix} pf(n^*(z;p)) \\ zf(n^*(z;p)) \end{pmatrix},$$

so that $\pi^*(\cdot)$ is strictly increasing in both of its arguments.

To show the final claim, note that, for any $z \in Z$,

$$\lim_{p \downarrow 0} (f')^{-1}\left(\frac{1}{pz}\right) = 0$$

by the Inada conditions, and as such

$$\lim_{p \downarrow 0} \pi^*(z;p) = -\phi.$$

$\pi^*(z;\cdot)$ is strictly increasing on $(0,+\infty)$, so this implies that

$$-\phi = \inf\{\pi^*(z;p) \mid p > 0\}.$$

Meanwhile, we can see that

$$\lim_{p \to \infty} (f')^{-1}\left(\frac{1}{pz}\right) = +\infty$$

by the Inada conditions, and by assumption,

$$pz\frac{f\left[(f')^{-1}\left(\frac{1}{pz}\right)\right]}{(f')^{-1}\left(\frac{1}{pz}\right)} = \frac{f\left[(f')^{-1}\left(\frac{1}{pz}\right)\right]}{f'\left((f')^{-1}\left(\frac{1}{pz}\right)\right)(f')^{-1}\left(\frac{1}{pz}\right)} \geq \alpha > 1$$

for any $p > 0$. Therefore,

$$\lim_{p \to \infty} \pi^*(z,p) = \lim_{p \to \infty} \left(\left[pz\frac{f\left[(f')^{-1}\left(\frac{1}{pz}\right)\right]}{(f')^{-1}\left(\frac{1}{pz}\right)} - 1\right](f')^{-1}\left(\frac{1}{pz}\right) - \phi\right) = +\infty.$$

Q.E.D.

Because the profit function $\pi^*(\cdot)$ is unbounded on $Z \times (0,+\infty)$, to apply our usual dynamic programming results we need to furnish an upper bound for the price p. We just showed above that

$$\pi^*(\underline{z};p) \to +\infty$$

as $p \to \infty$, and that $\pi^*(\underline{z};\cdot)$ is a strictly increasing function on $(0,+\infty)$. As such, there should exist some $\overline{p} > 0$ such that

$$\pi^*(\underline{z};\overline{p}) > \frac{\kappa}{\beta}.$$

We now restrict the domain of interest of the profit function $\pi^*(\cdot)$ to $Z \times (0,\overline{p}]$. We can immediately see that $\pi^*(\cdot)$ is bounded on $Z \times (0,\overline{p}]$, since

$$-\phi < \pi^*(z,p) \leq \pi^*(\underline{z};\overline{p})$$

for any $(z,p) \in Z \times (0,\overline{p}]$.

Denote $S = Z \times (0,\overline{p}]$ and let $\mathcal{S}$ be the corresponding product σ-algebra. Define the operator $T : C_b(S,\mathbb{R}) \to C_b(S,\mathbb{R})$ as

$$[T(f)](z,p) = \pi^*(z;p) + \beta \max\left[\int_Z f(z',p)P(z,dz'),\ 0\right]$$

for any $(z,p) \in S$ and $f \in C_b(S,\mathbb{R})$. Note that the operator $T(\cdot)$ does indeed map into the space of continuous and bounded functions, since

$$(z,p) \mapsto \int_Z f(z',p)P(z,dz')$$

is continuous on S by the extended Feller property, and boundedness is preserved under integrals with respect to probability measures. The following shows us that we can apply Banach's fixed point theorem to T:

Theorem (Existence of a Fixed Point)
Suppose that our model assumptions hold, and let $T(\cdot)$ be the operator on $C_b(S,\mathbb{R})$ defined above. Then, $T(\cdot)$ is a contraction mapping on the complete metric space $(C_b(S,\mathbb{R}), d_\infty)$, where d_∞ is the supremum metric on $C_b(S,\mathbb{R})$. As such, by Banach's fixed point theorem, there exists a unique fixed point $V^* \in C_b(S,\mathbb{R})$ such that $T(V^*) = V^*$, or equivalently,

$$V^*(z;p) = \pi^*(z;p) + \beta \cdot \max\left[\int_Z V^*(z';p)P(z,dz'), 0\right]$$

for any $(z,p) \in S$.

Proof) We verify that Blackwell's sufficient conditions holds for the operator $T(\cdot)$. First, choose any $f, g \in C_b(S, \mathbb{R})$ such that $f \leq g$ on S. Then, for any $(z, p) \in S$,

$$\int_Z f(z', p) P(z, dz') \leq \int_Z g(z', p) P(z, dz')$$

by the monotonicity of integration, so that

$$\begin{aligned}
[T(f)](z, p) &= \pi^*(z, p) + \beta \cdot \max\left[\int_Z f(z', p) P(z, dz'), 0\right] \\
&\leq \pi^*(z, p) + \beta \cdot \max\left[\int_Z g(z', p) P(z, dz'), 0\right] = [T(g)](z, p).
\end{aligned}$$

This proves that $T(f) \leq T(g)$ on S, so that $T(\cdot)$ satisfies the monotonicity condition.

Now choose any $f \in C_b(S, \mathbb{R})$ and $c \geq 0$. Then, for any $(z, p) \in S$,

$$\int_Z (f + c)(z', p) P(z, dz') = \int_Z f(z', p) P(z, dz') + c$$

because $P(z, \cdot)$ is a probability measure, and we have

$$\begin{aligned}
[T(f + c)](z, p) &= \pi^*(z, p) + \beta \cdot \max\left[\int_Z (f + c)(z', p) P(z, dz'), 0\right] \\
&= \pi^*(z, p) + \beta \cdot \max\left[\int_Z f(z', p) P(z, dz') + c, 0\right] \\
&\leq \pi^*(z, p) + \beta \left(\max\left[\int_Z f(z', p) P(z, dz'), 0\right] + c\right) = [T(f)](z, p) + \beta c,
\end{aligned}$$

where the inequality follows from the fact that $c \geq 0$. Therefore, $T(f + c) \leq T(f) + \beta c$, where $\beta \in (0, 1)$, and the discounting property is satisfied. It follows from Blackwell's sufficient conditions that $T(\cdot)$ is a contraction mapping on $(C_b(S, \mathbb{R}), d_\infty)$.

The existence of a unique fixed point $V^* \in C_b(S, \mathbb{R})$ now follows from an application of the Banach fixed point theorem.

Q.E.D.

Having established the existence of a unique continuous value function for the incumbent firm, we can further show that this value function is strictly monotonic in productivity and prices through an application of the properties theorem.

Theorem (Monotonicity of the Value Function)
Suppose that our model assumptions hold, and let $V^* : S \to \mathbb{R}$ be the value function found above. Then, $V^*(\cdot)$ is strictly increasing on S.

Proof) Define the subset C_1 of functions in $C_b(S,\mathbb{R})$ as follows:

$$C_1 = \{f \in C_b(S,\mathbb{R}) \mid f(\cdot,p) \text{ is increasing on } Z \ \forall p \in (0,\overline{p}]\}.$$

We can see that C_1 is a closed subset of $C_b(S,\mathbb{R})$; for any sequence $\{f_n\}_{n\in N_+}$ in C_1 that converges uniformly to some $f \in C_b(S,\mathbb{R})$,

$$f_n(z,p) \leq f_n(z',p)$$

for any $z, z' \in Z$ such that $z < z'$, $p \in (0,\overline{p}]$ and $n \in N_+$ by definition. Because uniform convergence implies pointwise convergence, taking $n \to \infty$ implies

$$f(z,a) \leq f(z',a),$$

so that $f \in C_1$. Now define

$$C_2 = \{f \in C_b(S,\mathbb{R}) \mid f(\cdot,p) \text{ is strictly increasing on } Z \ \forall p \in (0,\overline{p}]\}.$$

Choose some $f \in C_1$. We want to show that $T(f) \in C_2$; this will establish that $T(C_1) \subset C_2 \subset C_1$, and $V^* \in C_2$ by the properties theorem. Fix $p \in (0,\overline{p}]$ and $z, z' \in Z$ such that $z < z'$. Then, by the monotonicity of the kernel P,

$$\int_Z f(z_0,p)P(z,dz_0) \leq \int_Z f(z_0,p)P(z',dz_0),$$

since $f(\cdot,p)$ is increasing on Z by the fact that $f \in C_1$. As such,

$$\begin{aligned}[T(f)](z,p) &= \pi^*(z;p) + \beta \cdot \max\left[\int_Z f(z_0,p)P(z,dz_0), 0\right] \\ &\leq \pi^*(z;p) + \beta \cdot \max\left[\int_Z f(z_0,p)P(z',dz_0), 0\right] \\ &< \pi^*(z';p) + \beta \cdot \max\left[\int_Z f(z_0,p)P(z,dz_0), 0\right] = [T(f)](z',p),\end{aligned}$$

where the last inequality follows from the fact that $\pi^*(\cdot,p)$ is strictly increasing on Z. This implies that $T(f) \in C_2$, and therefore we can conclude that $V^*(\cdot,a)$ is strictly increasing on Z for any $p \in (0,\overline{p}]$.

The statements about p follows in an almost identical manner (except this time we do not require the monotonicity of the kernel P).

Q.E.D.

Having established the monotonicity of the value function, it is now very easy to show that a stationary equilibrium price exists.

Theorem (Existence of Stationary Equilibrium Price)
Suppose that our model assumptions hold, and let $V^* : S \to \mathbb{R}$ be the value function found above. Then, there exists a unique price $p^* \in (0,\overline{p}]$ that satisfies the free entry condition, that is,

$$-\kappa + \beta \int_Z V^*(z;p^*)d\gamma(z) = 0.$$

Proof) Define the function $g : (0,\overline{p}] \to \mathbb{R}$ as

$$g(p) = \int_Z V^*(z;p)d\gamma(z)$$

for any $p \in (0,\overline{p}]$. We can clearly see that $g(\cdot)$ is a strictly increasing function; for any $0 < p < p' \leq \overline{p}$, because $V^*(\cdot;p) < V^*(\cdot;p')$ by the strict monotonicity of

the value function, we have

$$g(p) = \int_Z V^*(z;p)d\gamma(z) \leq \int_Z V^*(z;p')d\gamma(z) = g(p').$$

If this inequality holds as an equality, then the vanishing property for non-negative functions implies that

$$V^*(z;p) = V^*(z;p')$$

for γ-a.e. $z \in Z$, which contradicts the fact that $V^*(\cdot;p) < V^*(\cdot;p')$ everywhere on Z. Thus, $g(p) < g(p')$.

We can also see that $g(\cdot)$ is a continuous function. Choose any sequence $\{p_n\}_{n\in N_+}$ in $(0,\overline{p}]$ that converges to $p \in (0,\overline{p}]$. Since convergent sequences are bounded, there exists some $p' \leq \overline{p}$ such that $p_n, p \leq p'$ for any $n \in N_+$. It follows that $\{V^*(\cdot;p_n)\}_{n\in N_+}$ is a sequence of continuous functions that is dominated by the γ-integrable function $V^*(\cdot;p')$ (by the monotonicity of the value function in prices) and which converges pointwise to $V^*(\cdot;p)$ (by the continuity of the value function). Therefore, by the DCT,

$$\lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} \int_Z V^*(\cdot;p_n)d\gamma = \int_Z V^*(\cdot;p)d\gamma = g(p).$$

This proves the continuity of $g(\cdot)$.

For any $p \in (0,\overline{p}]$, we know that

$$V^*(\overline{z};p) = \pi^*(\overline{z};p) + \beta \max\left[\int_Z V^*(z';p)P(\overline{z},dz'),0\right]$$

$$\leq \pi^*(\overline{z};p) + \beta \max[V^*(\overline{z};p),0]$$

by the strict monotonicity of $V^*(\cdot)$ in productivity. Because

$$\lim_{p\downarrow 0} \pi^*(\overline{z};p) = -\phi < 0,$$

there exists some $\underline{p} > 0$ such that $\pi^*(\overline{z};\underline{p}) < 0$. Suppose $V^*(\overline{z};\underline{p}) \geq 0$. Then,

$$V^*(\overline{z};\underline{p}) \leq \pi^*(\overline{z};\underline{p}) + \beta V^*(\overline{z};\underline{p}),$$

so that

$$V^*(\overline{z};\underline{p}) \leq \frac{1}{1-\beta}\pi^*(\overline{z};\underline{p}) < 0,$$

a contradiction. Therefore, it must be the case that $V^*(\overline{z};\underline{p}) < 0$ as well. Now we can see that

$$g(\underline{p}) = \int_Z V^*(z';\underline{p})d\gamma(z') \leq V^*(\overline{z};\underline{p}) < 0$$

as well.

Now it remains to show that $g(\overline{p}) > \frac{\kappa}{\beta}$. To this end, note that

$$\begin{aligned} g(\overline{p}) &= \int_Z V^*(z';\overline{p})d\gamma(z') \geq V^*(\underline{z};\overline{p}) \\ &= \pi^*(\underline{z};\overline{p}) + \beta \max\left[\int_Z V^*(z';\overline{p})P(\underline{z},dz'), 0\right] \\ &\geq \pi^*(\underline{z};\overline{p}) > \frac{\kappa}{\beta}, \end{aligned}$$

where the last inequality follows from the way we chose $\overline{p}$.

We have shown that $g(\cdot)$ is a continuous and strictly increasing function such that

$$g(\underline{p}) < 0 < \frac{\kappa}{\beta} < g(\overline{p}).$$

Therefore, by the intermediate value theorem, there exists some $p^* \in (\underline{p},\overline{p})$ such that $g(p^*) = 0$, that is,

$$\beta \int_Z V^*(z;p^*)d\gamma(z) = \kappa.$$

The uniqueness of p^* follows from the fact that $g(\cdot)$ is strictly increasing.

Q.E.D.

8.1.4 Characterizing the Exit Decision

The monotonicity of the value function in productivity, in particular, allows us to obtain a very simple characterization of the exit decision $\chi^*(\cdot;p^*): Z \to \mathbb{R}$ at equilibrium.

Theorem (Cutoff for Optimal Exit Decision at Equilibrium)
Suppose that our model assumptions hold, and let $V^*: S \to \mathbb{R}$ be the value function found above. Then, there exists some $z^* \in Z$ such that

$$z^* = \inf\left\{z \in Z \mid \int_Z V^*(z';p)P(z,dz') > 0\right\},$$

and for any $z \in Z$,

$$\chi^*(z;p^*) = \begin{cases} 1 & \text{if } z \leq z^* \\ 0 & \text{if } z > z^* \end{cases}.$$

is an optimal exit policy function at equilibrium.

Proof) We first prove that z^* is well-defined in Z. Because γ is the stationary distribution of P, for any increasing and bounded $\mathcal{Z}$-measurable function f, we have

$$\begin{aligned}\int_Z f(z)d\gamma(z) &= \int_Z \int_Z f(z')P(z,dz')d\gamma(z) \\ &\leq \int_Z \left(\int_Z f(z')P(\overline{z},dz')\right)d\gamma(z) = \int_Z f(z')P(\overline{z},dz'),\end{aligned}$$

where the inequality follows from the monotonicity of the kernel P. Since $V^*(\cdot;p^*)$ is a strictly increasing function on Z, it now follows that

$$0 < \frac{\kappa}{\beta} = \int_Z V^*(z';p^*)d\gamma(z') \leq \int_Z V^*(z';p^*)P(\overline{z},dz'),$$

where the first equality was established in the previous theorem. It follows that

$$\overline{z} \in \left\{ z \in Z \mid \int_Z V^*(z';p) P(z,dz') \geq 0 \right\},$$

so that the set on the right hand side is non-empty.

Define the function $h : Z \to \mathbb{R}$ as

$$h(z) = \int_Z V^*(z';p^*) P(z,dz')$$

for any $z \in Z$. By the continuity of $V^*(\cdot)$ and the Feller property of P, we can see that $h(\cdot)$ is a continuous function on Z. Furthermore, the monotonicity of P coupled with the fact that $V^*(\cdot;p^*)$ is strictly increasing on Z indicates that $h(\cdot)$ is also strictly increasing on Z. We have just shown that the set $\{z \in Z \mid h(z) \geq 0\}$ is non-empty. Since $h(\cdot)$ is continuous, and the inverse image of closed sets under continuous functions is also closed, we can see that this set is a closed subset of the compact set Z and therefore itself compact.

Since the infimum of a non-empty compact set is contained in the set itself, we can now see that there exists a $z^* \in Z$ such that $h(z^*) \geq 0$ and

$$z^* = \inf \left\{ z \in Z \mid \int_Z V^*(z';p) P(z,dz') \geq 0 \right\}.$$

Now let $z < z^*$. $h(z) \geq 0$ contradicts the fact that z^* is the infimum of the set $\{z' \in Z \mid h(z') > 0\}$, so we must have $h(z) < 0$. It follows that

$$\chi^*(z;p^*) = 1 \quad \text{if } z < z^*.$$

Meanwhile, suppose that $z > z^*$. Then, $h(z) > h(z^*) \geq 0$ by the strict monotonicity of z, meaning that

$$\chi^*(z;p^*) = 0 \quad \text{if } z > z^*.$$

It follows that

$$\chi^*(z;p^*) = \begin{cases} 1 & \text{if } z \leq z^* \\ 0 & \text{if } z > z^* \end{cases}$$

is an optimal exit decision function at equilibrium.

Q.E.D.

It follows that the stationarity condition can be reformulated as

$$\lambda(\cdot) = \int_{z^*}^{\overline{z}} P(z,\cdot) d\lambda(z) + E \cdot \gamma(\cdot).$$

Chapter 9

Search and Unemployment

9.1 The McCall Job Search Model

We first discuss the McCall job searching model. To begin with, we investigate how potential workers make decisions in the baseline model in which workers, once employed, are never let go. Then we extend this preliminary model into one with job separation, which allows us to model unemployment in the economy and also allows us to study steady state unemployment.

9.1.1 The Baseline McCall Model

The model assumptions are as follows:

Workers The economy is populated by many homogeneous households searching for a job, and a single firm that posts wages every period. Every household starts off the economy unemployed, and in each period gets to decide whether to accept or reject the wage that the firm posts. If they accept the wage offer at time t, then they become employed at time t and start receiving the offered wage right away, while if they reject the offer, then they receive unemployment benefits today and wait until the next period and decide once again whether to accept an offer tomorrow.

Compensation When unemployed, households receive fixed unemployment benefits of $b > 0$. Wage offers are drawn from the set $\mathcal{W} = [\underline{w}, \overline{w}] \subset \mathbb{R}_+$ according to some prob-

ability measure μ on $\mathcal{W}$. If a household accepts a wage offer of $w \in \mathcal{W}$, then it receives w forever.

Preferences We assume that each household has the same flow utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ given by $u(c) = c$ for any $c \in \mathbb{R}_+$. Households do not save and consume all of their income, so the flow utility of an unemployed household is b, while that of an employed household receiving wage $w \in \mathcal{W}$ is equal to w. The representative household's lifetime utility under the consumption path $\{c_t\}_{t \in \mathbb{N}}$ is

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$

where $\beta \in (0, 1)$ is the discount factor.

Instead of dealing with the household's sequence problem directly, we can see that it is easier to work directly with the recursive formulation of the problem. Specifically, in this model the value function of a worker at any time depends on two state variables: an indicator of employment e and, if this worker is employed, the wage he receives $w \in \mathcal{W}$.

Starting with the employed worker's problem, note that, if an employed worker receives wage $w \in \mathcal{W}$, then his discounted lifetime utility is given as

$$V_e(w) = \sum_{t=0}^{\infty} \beta^t \cdot w = \frac{w}{1-\beta}.$$

Moving onto the unemployed worker's problem, let $V_U(w)$ denote the lifetime utility of a worker that is unemployed at some point in time and receives a wage offer w. If the worker accepts the offer w, then he becomes employed right away and receives a lifetime utility of $V_e(w) = \frac{w}{1-\beta}$. On the other hand, if the worker rejects the offer, then he receives unemployment benefits b today and faces another wage offer tomorrow, drawn again from the distribution μ. Therefore, his expected discounted lifetime utilty from rejecting the offer w is given as

$$b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w').$$

It follows that the unemployed worker's recursive problem can be written as

$$V_U(w) = \max_{e \in \{0,1\}} \left[(1-e) \left(b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w') \right) + e \cdot \frac{w}{1-\beta} \right]$$

$$= \max \left[\frac{w}{1-\beta}, \; b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w') \right],$$

where $e = 0$ represents a rejection of the wage offer. Let the optimal acceptance policy $e^* : \mathcal{W} \to \{0,1\}$ be defined as a function such that

$$V_U(w) = (1 - e^*(w)) \left(b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w') \right) + e^*(w) \cdot \frac{w}{1-\beta}$$

for any $w \in \mathcal{W}$, or in other words,

$$e^*(w) = \begin{cases} 1 & \text{if } \frac{w}{1-\beta} \geq b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w') \\ 0 & \text{otherwise} \end{cases}.$$

In the McCall model, the optimal acceptance policy $e^* : \mathcal{W} \to \{0,1\}$ assumes a particularly simple form. Namely, the worker adopts a cutoff strategy, so that he accepts if and only if the offered wage is greater than or equal to some reservation wage. This is formally shown below:

Theorem (Optimal Acceptance Strategy)
Suppose that our model assumptions hold. Then, there exists a unique $w^* \in \mathcal{W}$ such that the optimal acceptance policy $e^* : \mathcal{W} \to \{0,1\}$ becomes

$$e^*(w) = \begin{cases} 1 & \text{if } w \geq w^* \\ 0 & \text{if } w < w^* \end{cases}.$$

Proof) By definition, $e^*(w) = 1$ if and only if $\frac{w}{1-\beta} \geq b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w')$. Furthermore, this holds if and only if

$$w \geq (1-\beta) \left(b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w') \right).$$

Denote the right hand side term by $\hat{w}$. If $\hat{w} < \underline{w}$, then let $w^* = \underline{w}$; if $\hat{w} > \overline{w}$, then let $w^* = \overline{w} + \epsilon$ for some $\epsilon > 0$; and if $\hat{w} \in \mathcal{W}$, then let $w^* = \hat{w}$.

If $\hat{w} < \underline{w}$, then because the expected payoff $\frac{w}{1-\beta}$ from accepting any wage offer $w \in \mathcal{W}$ is greater than the expected payoff from rejecting it, the worker accepts all wage offers, so that $e^*(w) = 1$, that is, the cutoff w^* is just the lowest wage $\underline{w}$. Meanwhile, if $\hat{w} > \overline{w}$, then because the worker does not accept any wage offers, $e^*(w) = 0$, or equivalently, the cutoff w^* is strictly greater than any wage in $\mathcal{W}$. Finally, if $\hat{w} \in \mathcal{W}$, then we can see that putting $w^* = \hat{w}$ shows us that the worker accepts the wage offer w if and only if $w \geq w^*$.

Q.E.D.

Having established the existence of a reservation wage w^* for the worker, it remains now to pin down its value. Suppose that $w^* \in \mathcal{W}$. Then, by the above theorem, we can see that

$$\frac{w^*}{1-\beta} = b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w') d\mu(w'),$$

and as such that

$$V_U(w) = \begin{cases} \frac{w}{1-\beta} & \text{if } w \geq w^* \\ \frac{w^*}{1-\beta} & \text{if } w < w^* \end{cases}.$$

In other words, the value function will be constant for all wages below w^*, and then it will be increasing in the wage level for any wages greater than or equal to w^*. Substituting this expression for the value function into the above equation, it turns out that w^* must satisfy

$$\frac{w^*}{1-\beta} = b + \beta \cdot \left(\int_{w^*}^{\overline{w}} \frac{w}{1-\beta} d\mu(w) + \frac{w^*}{1-\beta} \cdot \mu([\underline{w}, w^*)) \right).$$

It follows that we can further simplify the expression on the right hand side by noting that

$$\int_{w^*}^{\overline{w}} \frac{w - w^*}{1-\beta} d\mu(w) + \frac{w^*}{1-\beta} = \int_{w^*}^{\overline{w}} \frac{w}{1-\beta} d\mu(w) + \frac{w^*}{1-\beta} \left(1 - \mu([\underline{w}, w^*)^c)\right)$$

$$= \int_{w^*}^{\overline{w}} \frac{w}{1-\beta} d\mu(w) + \frac{w^*}{1-\beta} \cdot \mu([\underline{w}, w^*)).$$

Therefore,

$$\frac{w^*}{1-\beta} = b + \beta \cdot \int_{w^*}^{\overline{w}} \frac{w - w^*}{1-\beta} d\mu(w) + \beta \frac{w^*}{1-\beta},$$

rearranging which we obtain

$$w^* - b = \frac{\beta}{1-\beta} \cdot \int_{w^*}^{\overline{w}} (w - w^*) d\mu(w).$$

This is the core equation that determines the level of the worker's reservation wage. We can interpret the reservation wage w^* as the wage formed at the intersection of two curves: the cost $c(w) = w - b$ from rejecting the wage offer w and searching again, and the benefit $B(w) = \frac{\beta}{1-\beta} \cdot \int_w^{\overline{w}} (w' - w) d\mu(w')$ from waiting one period and receiving a higher offer than w. Naturally, the higher w, the lower the probability of receiving a higher offer than w tomrrow, so that $B(w)$ is decreasing in w. Clearly, the cost $c(w)$ is increasing in w. The reservation wage w^* is then the wage level that makes the worker indifferent between accepting the wage offer today and searching again tomorrow, that is, the wage level that equates the cost and benefit of searching.

Using this interpretation, we can study the comparative statics of the reservation wage w^*. Suppose unemployment benefits increase from b to b'. Then, the cost of searching decreases, that is, the curve $c(w)$ shifts downward, so that the worker is willing to hold out for another period given wage offers that he would have accepted before; in other words, the reservation wage w^* increases.

Now suppose that the discount factor β increases, that is, workers become more patient. Then, because $\frac{\beta}{1-\beta}$ increases, the benefit $B(w)$ to waiting for a higher wage offer increases, which means once again that the worker's reservation wage w^* rises.

9.1.2 The McCall Model with Job Separation

In the baseline McCall model, as long as $w^* < \overline{w}$, workers will eventually accept a wage offer, and since workers, once employed, are never let go, the unemployment rate of the economy converges to 0. This is clearly unrealistic and an undesirable feature of the model when we want to study the dynamcis of unemployment. As such, we consider a variant of the McCall model with exogenous job separation.

Formally, we augment the baseline McCall model with the following assumption:

Job Separation With probability $\alpha \in (0,1)$, an employed worker loses his job and becomes unemployed. If an employed worker is let go at time t, they spend period t as an unemployed worker receiving benefits b and start receiving wage offers starting from time $t+1$. We continue to assume that a worker who accepts a wage offer $w \in \mathcal{W}$ continues to receive that wage until he is let go.

This seemingly innocuous assumption is enough to completely change the McCall model's implications on unemployment. Let $V_e(w)$ be the value function of an employed worker who receives wage w, $\bar{V}_U$ be the value function of an unemployed worker who starts receiving job offers starting from the next period, and $V_U(w)$ the value function of an unemployed worker who receives the wage offer $w \in \mathcal{W}$.

We can see that $V_e(w)$ can be expressed as the sum of today's flow utility and tomorrow's expected discounted lifetime utility. First, the worker receives wage w today. Tomorrow, he is let go with probability α and receives a wage offer, at which point his expected lifetime utility becomes $\bar{V}_U$, since he can only start receiving wage offers the day after tomorrow. On the other hand, with probability $1-\alpha$ the worker retains his job and continues to receive the wage w, so that his lifetime utility continues to be $V_e(w)$. Putting these elements together, the employed worker's recursive problem is given by

$$V_e(w) = w + \beta\alpha \cdot \bar{V}_U + \beta(1-\alpha) \cdot V_e(w).$$

Rearranging this so that the $V_e(w)$ terms are on the left hand side shows us that

$$V_e(w) = \frac{w}{1-\beta(1-\alpha)} + \frac{\beta\alpha}{1-\beta(1-\alpha)} \cdot \bar{V}_U.$$

Now consider an unemployed worker who receives the wage offer $w \in \mathcal{W}$ today. As

in the baseline model, the value function $V_U(w)$ of this worker is given as

$$V_U(w) = \max\left[V_e(w),\ b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w')d\mu(w')\right],$$

with the optimal acceptance policy $e^* : W \to \{0, 1\}$ satisfying

$$V_U(w) = e^*(w)V_e(w) + (1 - e^*(w))\left[b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w')d\mu(w')\right].$$

for any $w \in \mathcal{W}$.

Finally, consider an unemployed worker who starts receiving wage offers tomorrow. His flow utility today is equal to b, the size of the unemployment benefits, while his expected lifetime utility starting tomorrow is equal to $\int_{\underline{w}}^{\overline{w}} V_U(w')d\mu(w')$. Therefore,

$$\bar{V}_U = b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w')d\mu(w').$$

We currently have three kinds of value functions that characterize the dynamics of the model. To simplify the model further, note that

$$\begin{aligned}\int_{\underline{w}}^{\overline{w}} V_U(w)d\mu(w') &= \int_{\underline{w}}^{\overline{w}} \max\left[V_e(w),\ b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w')d\mu(w')\right] d\mu(w) \\ &= \int_{\underline{w}}^{\overline{w}} \max\left[V_e(w),\ \bar{V}_U\right] d\mu(w).\end{aligned}$$

This implies that $\bar{V}_U$ satisfies the equation

$$\bar{V}_U = b + \beta \cdot \int_{\underline{w}}^{\overline{w}} \max\left[V_e(w),\ \bar{V}_U\right] d\mu(w),$$

so that we have reduced the model from one involving $V_e(w), V_U(w)$ and $\bar{V}_U$ to one that involves only $V_e(w)$ and $\bar{V}_U$.

Recall that the optimal acceptance policy e^* is determined so that $e^*(w) = 1$ if and

only if

$$V_e(w) \geq b + \beta \cdot \int_{\underline{w}}^{\overline{w}} V_U(w) d\mu(w') = \bar{V}_U.$$

Using the expression for $V_e(w)$ obtained above, we can see that $e^*(w) = 1$ if and only if

$$\frac{w}{1-\beta(1-\alpha)} + \frac{\beta\alpha}{1-\beta(1-\alpha)} \cdot \bar{V}_U \geq \bar{V}_U,$$

or equivalently,

$$w \geq (1-\beta)\bar{V}_U.$$

Therefore, assuming that $(1-\beta)\bar{V}_U \in \mathcal{W}$, we can see that

$$w^* = (1-\beta)\bar{V}_U$$

is the unemployed worker's reservation wage, with the optimal acceptance policy assuming the cutoff form

$$e^*(w) = \begin{cases} 1 & \text{if } w \geq w^* \\ 0 & \text{if } w < w^* \end{cases}.$$

To once again search for the equilibrium equation that determines the reservation wage, note first that, since $V_e(w) \geq \bar{V}_U$ if and only if $w \geq w^*$,

$$\begin{aligned} \bar{V}_U &= b + \beta \cdot \int_{\underline{w}}^{\overline{w}} \max\left[V_e(w),\, \bar{V}_U\right] d\mu(w) \\ &= b + \beta \left[\int_{w^*}^{\overline{w}} V_e(w) d\mu(w) + \bar{V}_U \cdot \mu([\underline{w}, w^*)) \right], \end{aligned}$$

as in the baseline model. In addition, $\bar{V}_U = \frac{w^*}{1-\beta}$ and

$$V_e(w) = \frac{w}{1-\beta(1-\alpha)} + \frac{\beta\alpha}{1-\beta(1-\alpha)} \cdot \frac{w^*}{1-\beta}$$

for any $w \in \mathcal{W}$, we have

$$\begin{aligned}\frac{w^*}{1-\beta} &= b + \beta \cdot \int_{w^*}^{\overline{w}} \left[\frac{w}{1-\beta(1-\alpha)} + \frac{\beta\alpha}{1-\beta(1-\alpha)} \cdot \frac{w^*}{1-\beta} \right] d\mu(w) + \beta \cdot \frac{w^*}{1-\beta} \cdot \mu([\underline{w}, w^*)) \\ &= b + \frac{\beta}{1-\beta} w^* + \beta \cdot \int_{w^*}^{\overline{w}} \left[\frac{w}{1-\beta(1-\alpha)} + \left(\frac{\beta\alpha}{1-\beta(1-\alpha)} - 1 \right) \frac{w^*}{1-\beta} \right] d\mu(w) \\ &= b + \frac{\beta}{1-\beta} w^* + \beta \cdot \int_{w^*}^{\overline{w}} \frac{w - w^*}{1-\beta(1-\alpha)} d\mu(w).\end{aligned}$$

It now follows that

$$w^* - b = \frac{\beta}{1-\beta(1-\alpha)} \cdot \int_{w^*}^{\overline{w}} (w - w^*) d\mu(w).$$

Note that the only difference between this equation and the corresponding equation in the baseline McCall model is the smaller coefficient $\frac{\beta}{1-\beta(1-\alpha)}$ for $\int_{w^*}^{\overline{w}} (w - w^*) d\mu(w)$. This takes into consideration the fact that, even when workers accept a job offer, they can be let go in the future, so that the expected benefits from continuing to search for a better job offer is lower in the model with job separation compared to the baseline model.

Let us now focus on the second object of interest, the unemployment rate. Let U_t be the unemployment rate at time t, that is, the proportion of workers who are unemployed at time t. Letting $F : \mathcal{W} \to [0,1]$ be the distribution function corresponding to the wage distribution μ, note that, at any time t, there are two distinct flows of workers. First, a proportion α of employed workers become unemployed. On the other hand, a proportion $1 - F(w^*)$ of unemployed workers accept their job offers and become employed. Therefore, the dynamics of unemployment in this economy is given by

$$U_{t+1} - U_t = \underbrace{\alpha(1 - U_t)}_{\text{Increase in UR}} - \underbrace{(1 - F(w^*)) \cdot U_t}_{\text{Decrease in UR}}.$$

Consider a steady state unemployment rate $U^* \in [0,1]$ such that $U_{t+1} = U_t = U^*$ at time t. Then, U^* satisfies

$$\alpha(1 - U^*) = (1 - F(w^*))U^*,$$

which we can solve for U^* as

$$U^* = \frac{\alpha}{\alpha + 1 - F(w^*)}.$$

We can see that increasing the job separation rate α naturally raises the steady state level of unemployment.

Furthermore, we would expect U^* to rise in response to an increase in b, since we expect greater unemployment benefits to induce workers to stay unemployed for longer. To see that this is indeed the case, return to the equation

$$w^* - b = \frac{\beta}{1 - \beta(1-\alpha)} \cdot \int_{w^*}^{\overline{w}} (w - w^*) d\mu(w).$$

characterizing the reservation wage. We can see that raising b lowers the cost of job searching, which means workers take on a higher reservation wage w^*; this in turn means that the proportion of unemployed workers who find new jobs at time t, $(1 - F(w^*))U_t$, decreases. Therefore, U^* ends up increasing, since its denominator decreases.